A Descriptive Model for Perception of Optical Illusions

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The apparent curves of stable geometrical illusions of angle are modeled by a first-order differential equation depending on a single parameter, called the strength of the illusion. The model is based on Brentano's theory that the human visual system tends to overestimate acute angles. It is mathematically equivalent to Hoffman's Lie-theoretic model (SIAM Review 13 (1971), pp. 169-184) except for some deficiencies of detail in Hoffman's approach which are noted and corrected. By reversing the sign of the strength parameter, "correction" curves may be plotted which cause the illusion to "disappear," thereby demonstrating the effectiveness and accuracy of the model. The method is illustrated with the classical illusions of Poggendorff, Zöllner, Hering, Orbison, Ponzo, and Müller-Lyer.

1. INTRODUCTION

Stable geometrical illusions may be roughly divided into two classes (Boring, 1942, p. 243; Teuber, 1960, p. 1654), namely illusions of angle (e.g., the Hering illusion, in which a straight line appears bowed by a distortion pattern) and illusions of extent (e.g., a vertical segment appearing longer than a horizontal one of the same size), but with some overlapping of the two classes, as represented by apparent changes of extent caused by angular relationships (e.g., the Müller-Lyer "arrowhead" illusion). This paper will consider only illusions of angle, but with that phrase specifically intended to include the "mixed" type of angular-induced illusions of extent. Our objective is to present a simple, first-order model which applies to such illusions, and which describes qualitatively and (in a limited sense) quantitatively what people typically see in these illusions.

A question of considerable importance is whether optical illusions occur because of the physiology of the retina or because of the neural processing of data by the visual cortex, or perhaps because of some combination of the two factors. The descriptive model presented here is displayed in terms of the geometry of the plane in which the illusion is drawn, and therefore can make no distinction between retinal and cortical theories. It can, however, contribute something (in a negative sense) to the understanding of this controversial point. The model itself is mathematically equivalent to that of Hoffman (1971) (except for some necessary adjustments to be noted below), which is based on a cortical theory, and it is consistent with the much more elaborate model of Walker (1973), which is based directly on the physiology of the retina. (See also MacLeod, Virsu, & Carpenter, 1974; Walker, 1974.) What this demonstrates is that the controversy...
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cannot be resolved solely by appeal to goodness of fit with observable data without consideration of more subtle, higher-order effects.

The noted historian of psychology, E. G. Boring (1942), describes a decade of remarkable activity in the 1890s which produced no less than a dozen distinct theories to explain illusions such as that of Müller–Lyer. He classifies these as seven “total impression” theories, three “dynamic” theories, and two “special” theories, not otherwise classified. (See Titchener, 1901, for further details of all twelve theories.) Of these, the two “special” theories have a rather special interest for describing the context of present model. One of these, due to Thiéry (1896), is based on three-dimensional perspective applied (or misapplied) to two-dimensional drawings. Tausch (1954) attempted to carry Thiéry’s idea to its logical conclusion by finding a prototypical “real life” constancy for every illusion, such as viewing the sides of the Ponzo illusion as sides of a long bridge, thereby introducing depth perception. Gregory (1968) noted that many plane illusions really do appear flat to most observers, and attempted to base an acceptable theory on Thiéry’s work while overcoming this basic difficulty by introducing unconsciously perceived depth. Perhaps the best that can be said of this was stated by Humphrey and Morgan (1965) (also quoted by Robinson, 1968, and Walker, 1973): “A theory which appeals to the idea of automatic compensation for unconsciously perceived depth is in obvious danger of being irrefutable.”

The other “special” theory of the 1890s was that of Brentano (1892, 1893), which we state formally for future reference.

Brentano’s Hypothesis. The human visual system tends to overestimate acute angles and to underestimate obtuse angles.

As Gregory states, this was a “supposed general principle ... although why this should be so was left unexplained.” It is the thesis of this paper that Brentano’s Hypothesis alone suffices to build an adequate descriptive model for a broad range of optical illusions, and that therefore the question of “why this should be so” deserves much more attention than it has been given. Furthermore, the misperception of angles is, as we will show, quite systematic and can be described in terms of a single parameter, the magnitude of which may be experimentally determined. Incidentally, Teuber’s explanation (1960, pp. 1656–1657) of Tausch’s work makes it clear that Brentano’s Hypothesis is one possible interpretation of his observations.

One other line of thought needs to be described as background for a model based on Brentano’s Hypothesis. Orbison (1939) based a study of the same optical illusions considered here on a quasi-physical theory attributed to Brown and Voth (1937), in which were postulated “field forces” analogous to magnetic or electric fields. The forces were of two types, “cohesive” and “restraining,” and a stable optical illusion was defined as one in which the vector sum of the cohesive forces exactly equals that of the restraining forces. (Brown and Voth had introduced the idea with a study of one of two possible unstable situations.) From the stability condition Orbison derived qualitative predictions about the apparent shapes of various figures when superimposed on various distortion fields, and experiments with approximately 30 subjects yielded unanimous agreement
with his predicted shapes. This, he concluded, confirmed "the hypothesis that the optic cortex functions as a vector-field," and pointed toward the possibility of a more general and quantitative theory of visual perception. Thus, the work of Orbison, Brown, and Voth may be seen as an intellectual precursor of Hoffman's Lie-theoretic approach (1971) to visual perception and to illusions in particular, about which we will have more to say later.

Of course, there are no physical "forces" at work in a geometrical drawing in the plane, but there certainly is a vector field associated with any distortion pattern. The appropriate coupling of that vector field with Brentano's Hypothesis is the key to a much simpler description of the observed distortions than that given by Orbinson, or for that matter, by Hoffman.

2. Simple Straight-Line Illusions

We will present the Brentano model first in the simplest possible context, namely for illusions in which all the "curves" are and appear to be straight lines, and in which the distortion pattern affects the perceived slope of one or more lines. The prototypes for such illusions are the Poggendorff and Zöllner illusions (Figs. 1 and 2, respectively). In the Poggendorff figure the distortion pattern consists of two parallel lines, and the distorted "curve" is the single sloping transversal, which appears to be parts of two different lines. In the Zöllner figure, we have the "dual" situation: The short segments with alternately positive and negative slopes form the distortion pattern, and the vertical parallels are distorted in that they appear to slope in alternate directions.

Observe that the apparent "offset" in the Poggendorff figure could arise either as linear shifts of the transversal line in opposite directions along the distortion lines or as enlarge-

![Fig. 1. The Poggendorff illusion.](image-url)
ment of the acute angles of intersection, i.e., the Brentano Hypothesis. Both explanations have been advanced. For example, Hoffman (1971) prefers the former explanation (although the latter is also consistent with his model), and Walker's model (1973) required the latter explanation. A convenient way to quantify the Brentano Hypothesis is illustrated in Fig. 3. Let $T$ be a unit vector in the direction of the distortion pattern, and $N$ a unit vector normal to the distorted curve at a point of intersection. Then an apparent normal $N_a$ is formed by adding a small multiple of $T$ to $N$:

$$N_a = N + \epsilon T,$$

(1)

where $\epsilon$ is a small positive constant we call the strength\(^1\) of the illusion or of the distortion pattern. The apparent slope at the point of intersection is then the slope of the line normal to $N_a$. (There are, of course, two possible choices for each of $T$ and $N$. Either choice for $T$ is permissible, but then $N$ must be chosen to form an acute angle with $T$, so that the apparent slope determined by Eq. (1) is consistent with Brentano's Hypothesis. Note that the apparent slope is independent of the choice made for $T$.)

\(^1\) This term is used informally for the single, abstract parameter appearing in the qualitative model under discussion, and it is not to be confused with other uses of the term in the psychological literature.
For the Poggendorff illusion, we may take \( T = (0, 1) \) and \( N = (m^2 + 1)^{1/2} (-m, 1) \), where \( m \) is the slope of the transversal. Then the apparent slope determined by (1) is

\[
m_a = \frac{m}{1 + \epsilon(m^2 + 1)^{1/2}}.
\]  

In order to determine the apparent lines in Fig. 1, we fix those points in the figure least subject to the distortion effect, i.e., the end points of the transversal. Thus, if these points are at \((\pm b, \pm mb)\), then the two apparent lines are given by

\[
y \mp mb = m_a(x \mp b).
\]

These lines are shown in Fig. 4, corresponding to Fig. 1 (for which \( m = 1 \)), with \( \epsilon = 0.035 \). The vertical separation between the two lines indicated by (3) is easily seen to be

\[
\Delta y = \frac{2mb}{1 + \frac{1}{\epsilon(m^2 + 1)^{1/2}}},
\]

which is 0 when \( m = 0 \) and increases with increasing \( m \) as it should.

The referee has noted that our choice of fixing end points of the transversal to plot the apparent lines leads to dependence of \( \Delta y \) on the length of the transversal. It is well known that the distortion \( \Delta y \) decreases with increasing length of the transversal (Robinson, 1968). However, the dependence in the present model is in the opposite direction. A simple computation with similar triangles shows that, if \( L \) is the length of each half-transversal, and each is increased by \( \Delta L \), then \( \Delta y \) increases by \( (\Delta y)(\Delta L)/L \). For small changes in length, this change is small relative to \( \Delta y \), therefore imperceptible.
However, this observation emphasizes the first-order nature of the model and the fact that it describes only the dominant angular distortion effect. Further comments regarding the analysis of the Poggendorff illusion appear in Section 5 below.

Both Brentano’s Hypothesis and the resulting model (1) are merely qualitative statements, of course, and therefore give no clue as to the magnitude of the strength parameter \( \epsilon \). Examination of pictures like Fig. 4 is not very helpful for this purpose, since perceived differences are very slight over considerable ranges for \( \epsilon \). A more fruitful approach is to reverse the distortion effect and “correct” the illusion by appropriate adjustment of \( \epsilon \). This is accomplished by defining a corrected normal

\[
N_c = N - \epsilon T
\]  

and drawing corresponding lines orthogonal to \( N_c \) on the original distortion pattern. Such a corrected Poggendorff illusion is shown in Fig. 5, again with \( \epsilon = 0.035 \) and end points of the original transversals fixed. It is obviously easier for an observer to decide whether the transversal in Fig. 5 does or does not appear to be a single straight line than it is for him to decide whether Fig. 4 appears to match Fig. 1.

Some comment is obviously necessary about how a choice of \( \epsilon \) was arrived at. For each of the illusions to be considered here (and a number of others as well), apparent curves and/or corrected illusions were drawn by computer plotter for \( \epsilon \) ranging from 0 to about 0.1 in steps of 0.01. These were shown to an unscientific sample of acquaintances for opinions about matching (in the case of apparent curves) or proper “correction.” The relatively high degree of consensus resulting from these informal “experiments” suggests that, for each illusion, \( \epsilon \) should be relatively constant across observers, and therefore readily determined by proper experiments. More remarkable was the fact that the
numbers 0.03 and 0.04 were selected as the appropriate strengths over a wide range of different illusions, suggesting the possibility that there really is a systematic Brentano effect that is independent of the illusion under consideration. More will be said later about conditions under which \( \epsilon \) can or cannot be expected to be constant.

Let us consider now the Zöllner illusion (Fig. 2). In this case the vector \( \mathbf{T} \) in the direction of the distortion pattern is \( \mathbf{T} = (m^2 + 1)^{-1/2} (1, m) \), where \( m \) is the common slope for one portion of the pattern, positive for the first and third parts, negative for the second and fourth parts. The unit normal to the distorted verticals making an acute angle with \( \mathbf{T} \) is \( \mathbf{N} = (1, 0) \). The apparent normal is then

\[
\mathbf{N}_a = \frac{1}{(m^2 + 1)^{1/2}} (\epsilon + (m^2 + 1)^{1/2}, \epsilon m),
\]

which leads to an apparent slope of

\[
m_a = -\frac{\epsilon + (m^2 + 1)^{1/2}}{\epsilon m}.
\]

Note that \( m_a \) is large in magnitude and opposite in sign to \( m \). Figure 6 shows the corrected Zöllner illusion with \( \epsilon = 0.02 \). In this case the midpoints of the original verticals have been left fixed and the lines have been drawn with the corrected slope

\[
m_e = \frac{(m^2 + 1)^{1/2} - \epsilon}{\epsilon m}.
\]

\(^2\) The choice of fixed points seems not to matter much for this correction. We have also plotted corrections with the end points at one end fixed, with essentially the same results.
If one looks first at the top ends of the "verticals" in Fig. 6 and then at the bottom ends, one may deduce that they are not in fact parallel. However, if one's gaze is directed at the center of the figure, there should be an illusion of parallelism.

Fig. 6. A corrected Zöllner illusion, \( \varepsilon = 0.02 \).

3. Illusions of Curvature

Many of the classical illusions of angle utilize a distortion pattern to make straight lines appear curved or circles out-of-round. In principle, a distortion pattern should distort any curve passing through it, but in order to see an illusion, one must know the "true" shape of the curve. For simplicity of presentation, and to draw on the most striking examples, we will illustrate the extension of the Brentano model to this case with two "bent line" illusions, the Orbison square illusion (Fig. 7a, 1939) and the Hering illusion (Fig. 8a, 1861). However, the model is capable of describing other distortions as well.

Suppose that the distorted curve(s) in an illusion are described parametrically (with respect to any convenient parameter) by a system of differential equations:

\[
x' = f(x, y),
\]

\[
y' = g(x, y).
\]

(9)
Then a unit normal at any point \((x, y)\) is given by

\[
N = \frac{1}{\left((x')^2 + (y')^2\right)^{1/2}} (-y', x'),
\]

(10)

with \(x'\) and \(y'\) given by (9). Similarly, if the curves of the distortion pattern are given parametrically by

\[
x' = h(x, y),
\]
\[
y' = k(x, y),
\]

(11)

then a unit tangent vector at any point \((x, y)\) (in particular, at any point of intersection with a distorted curve) is given by

\[
T = \frac{1}{\left((x')^2 + (y')^2\right)^{1/2}} (x', y'),
\]

(12)

with \(x'\) and \(y'\) determined by (11). Note that normalization of the vectors involved is equivalent to reparameterizing by arc length, which is appropriate for discussions of shape and curvature. At each intersection point of distorted with distorting curve, Brentano’s Hypothesis implies an apparent normal given by Eq. (1). As a function of position \((x, y)\) in the plane, we have

\[
N_b = \left(-\frac{g}{(f^2 + g^2)^{1/2}} + \frac{\epsilon h}{(h^2 + k^2)^{1/2}} + \frac{f}{(f^2 + g^2)^{1/2}} + \frac{ek}{(h^2 + k^2)^{1/2}}\right).
\]

(13)

To avoid possible difficulties in determining whether \(T\) and \(N\) form an acute angle for each possible choice of \((x, y)\), we will choose the sign of \(\epsilon\) so that \(N\) and \(\epsilon T\) form an acute angle, which is clearly all that is required by Brentano’s Hypothesis. The strength of the distortion pattern may then be redefined to be \(|\epsilon|\).

There are, of course, only a finite number of intersection points in any optical illusion. However, in order to have an illusion, the distortion pattern must be sufficiently “dense” that the intersection points occur fairly close together. Therefore it is appropriate to use a continuous model for this inherently discrete situation\(^3\) and conclude from (13) that the slope at any point along the distorted curve is given by

\[
\frac{dy}{dx} = \frac{g(h^2 + k^2)^{1/2} - \epsilon h(f^2 + g^2)^{1/2}}{f(h^2 + k^2)^{1/2} + \epsilon k(f^2 + g^2)^{1/2}}.
\]

(14)

One may also argue that the brain, presented with the conflicting data of apparent slopes given by (14) at a discrete set of points, plus an apparently smooth curve, automatically carries out just such a modeling process in order to resolve the conflict.

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\(^3\) The continuous model assumes a crossing point at every point of the distorted curve, i.e., that the distortion pattern is infinitely dense. Circumstances limiting the appropriateness of this modeling step are discussed below in Section 5.
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Thus the apparent curve arising from the distortion is determined as one of the solutions of a differential equation (14) of first order. In order to determine which solution, an appropriate initial condition must be assumed, and as in Section 2, we choose an appropriate point on an apparently undistorted part of the curve. Also as in Section 2, we assume that the strength parameter $|\epsilon|$ is constant for a given illusion, although that is not required by the model. As before, it is easier to verify the model qualitatively and to determine $|\epsilon|$ quantitatively by working with corrected illusions rather than apparent curves. The appropriate correction curve in each case is obtained by changing the sign of $\epsilon$ in Eq. (14).

![Fig. 7. (a) The Orbison square illusion. (b) Corrected illusion, $|\epsilon| = 0.035$.](image)

To illustrate the model, consider first the Orbison square illusion (Fig. 7a). It is sufficient, of course, to determine the apparent shape of one of the four sides of the square, say the upper horizontal, which is given parametrically by

$$x' = 1, \quad y' = 0. \quad (15)$$

Choosing the origin of the coordinate system at the center of the figure, the distortion pattern of concentric circles consists of solutions of

$$x' = -y, \quad y' = x. \quad (16)$$

Substitution of $f$ and $g$ from (15) and $h$ and $k$ from (16) into (14) yields

$$\frac{dy}{dx} = \frac{\epsilon y}{(x^2 + y^2)^{1/2} + \epsilon x}. \quad (17)$$

Since the normal $\mathbf{N}$ determined by (15) always points upward, and the tangent $\mathbf{T}$ determined by (16) points in the counterclockwise direction, we have

$$\text{sgn}(\epsilon) = \text{sgn}(x). \quad (18)$$
Thus (17) is really two different differential equations, one for the portion of the apparent curve on each side of the \(y\)-axis. Note that there is a discontinuity in the slope at \(x = 0\), as \(dy/dx\) changes abruptly from \(-|\epsilon|\) to \(|\epsilon|\). By symmetry, it is sufficient to solve either equation and reflect in the \(y\)-axis to get the other solution.

A standard polar coordinate substitution transforms Eq. (17) into

\[
\frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{\epsilon \sin \theta}{1 + \epsilon \cos \theta}.
\]

After multiplying out fractions and simplifying, (19) reduces to

\[
r' \sin \theta + r \cos \theta + \epsilon r = 0.
\]

Equation (20) is variable separable, and its solution is easily found to be

\[
r = K \csc \theta (\csc \theta + \cot \theta)^{\epsilon},
\]

where the constant \(K\) of integration is the value of \(y\) when \(\theta = \pi/2\) (or \(x = 0\)), say \(y_0\).

An equivalent form for (21) is

\[
y = y_0 \left( \frac{1 + \cos \theta}{\sin \theta} \right)^{\epsilon},
\]

or in terms of rectangular coordinates alone,

\[
y^{1+\epsilon} = y_0 ((x^2 + y^2)^{1/2} + x)^{\epsilon}, \quad \text{sgn}(\epsilon) = \text{sgn}(x).
\]

(Note that \(y_0\) need not be the \(y\)-coordinate on the original horizontal line, since the apparent distortion of \(y\) is maximal at \(x = 0\).)

One may verify that Eq. (23) has appropriate qualitative features for the apparent curve in the Orbison illusion, but it is much more convenient to plot the apparent curve directly from (17), using a numerical solution, or better yet, to reverse the sign of \(\epsilon\) and plot a correction curve on the original distortion pattern. Such a corrected illusion is shown in Fig. 7b, with \(|\epsilon| = 0.035\). The upper left-hand corner of the square was taken as initial point, the left-hand half of the upper curve plotted as a Runge–Kutta solution to (17), and the rest of the “square” drawn by symmetry. It may seem inappropriate that the apparent curve given by (23) is concave downward and has discontinuous slope at \(x = 0\). Nevertheless, the corrected illusion should be convincing evidence that these qualitative features are correct. The upward concavity and discontinuous slope of the corrected curve may be seen by holding the page at eye level and sighting along one of these curves.

The treatment of the Hering illusion (Fig. 8a) is similar. One of the horizontals (the upper one, say) is parametrized as in (15). A convenient parameterization for the distortion pattern of rays through the origin is given by

\[
x' = x, \quad y' = y.
\]
Then substitution into Eq. (14) yields the differential equation

\[
\frac{dy}{dx} = -\epsilon x \left( x^2 + y^2 \right)^{1/2} + \epsilon y
\]  

for the apparent curve. Since \( T \) determined by (24) is directed outward from the origin, it makes an acute angle with \( \mathbf{N} = (0, 1) \) at every point in the upper half plane, so \( \epsilon > 0 \) in (25). Reversing the sign of \( \epsilon \) produces the differential equation for the correction curve, and a plotted numerical solution of this equation with \( |\epsilon| = 0.035 \) is shown in Fig. 8b.\(^4\) (The initial point was chosen near the circle, with the line segments outside the circle left straight.)

As in the Orbison case, a closed-form solution of (25) may be found by polar coordinate substitution. After simplification, the equation becomes

\[
r'(\epsilon + \sin \theta) + r \cos \theta = 0. \tag{26}
\]

This equation is variable separable, and its solution is

\[
r = \frac{y_0(1 + \epsilon)}{\epsilon + \sin \theta}, \tag{27}
\]

where, as before, \( y_0 \) is the y-intercept of the apparent curve. Conversion of (27) back to rectangular coordinates yields

\[
y + \epsilon(x^2 + y^2)^{1/2} = y_0(1 + \epsilon). \tag{28}
\]

\(^4\) Compare Walker (1973, Fig. A-2).
4. Angular Illusions of Extent

The classical prototypes for illusions of extent that are apparently created by angular relationships are the Ponzo illusion (1928) and the Müller–Lyer illusion (1889), shown in Figs. 9a and 10a, respectively. Our version of the Ponzo illusion is adapted from Teuber (1960, p. 1657), where it is noted that the effect (i.e., the lengthening of the top of the square relative to the bottom) is similar to the effects noted by Orbison and Hering in patterns of converging rays. The connection between the Müller–Lyer illusion and other illusions of angle, such as the Poggendorff illusion, has been noted by Walker (1973, p. 478).

The Ponzo illusion has often been "explained" in terms of depth perception (Tausch, 1954; Teuber, 1960; Gregory, 1968), with the sloping lines (perhaps unconsciously) seen as parallels receding into the distance. However, if Fig. 9a really were a perspective drawing of a three-dimensional scene, the apparent difference in size between the "near" and "far" horizontal segments would be much greater than it seems. In fact, when perspective drawings of "real" scenes are used as background, the distortion effect is greatly enhanced (see Gregory, 1968, and Teuber, 1960, for such illustrations). Nevertheless, a significant residual effect remains in the stylized plane drawing shown here, devoid of any "real" context, and indeed, the effect is still present if viewed in other orientations, such as upside down. Furthermore, the effect is essentially the same if the middle five lines in Fig. 9a are deleted. On the other hand, if the figure is drawn with only the outermost sloping lines, the effect is greatly reduced, even though the top of the square still appears to be slightly longer. This suggests that the determining factors in the strength of the illusion are the angular crossings of sloping lines with the sides of the square, whether those sides are actually drawn in or not. Viewed this way, the Ponzo illusion is seen to be equivalent to the Hering illusion (Fig. 8) with the parallels vertical rather than horizontal, and with the "center" shifted off the top of the figure. The apparent difference in the lengths of the horizontal segments is a difference in distance between
different parts of apparent Hering curves. Essentially the same analysis that led to Fig. 8b was used for drawing the corrected Ponzo illusion in Fig. 9b. Of course, the angular crossings with the horizontals should also produce a bending effect, but the strength of that effect is small enough to be negligible, and it has in fact been neglected. The bottom corners of the square were taken as initial points, and $|\epsilon| = 0.035$ was used as in three of the four previous corrections.

![Diagram](a)

![Diagram](b)

**Fig. 10.** (a) The Müller-Lyer illusion. (b) Corrected illusion, $|\epsilon| = 0.14$.

As noted by Walker (1973), each of the angles in the Müller-Lyer illusion (Fig. 10a) may be treated as each of the angles in the Poggendorff illusion (Fig. 1) was. Except for orientation (interchanging horizontal and vertical, etc.), the apparent slopes are then given by formulas like (2). As in the Poggendorff case, the ends of the “wings” are held fixed (undistorted), lines with the apparent slopes are perceived, and the figure is completed by joining the intersections of the “wings,” thus producing an apparent “stretching” of one of the horizontal segments and “contraction” of the other. The corrected illusion is produced in the same way, reversing the sign of $\epsilon$ in each case. Such a corrected Müller-Lyer illusion is shown in Fig. 10b, with $|\epsilon| = 0.14$.

There is a very simple reason why the “strength” of this illusion is roughly four times that of the other illusions considered above. First of all, it is easy to see that having “wings” on both sides of the horizontal has a doubling effect on the strength. The reader may verify this informally by copying Fig. 10a without the lower wings. The illusion remains, but is clearly weaker. A value of $|\epsilon| = 0.07$ has been found to produce a good correction. Second, the Brentano effect produces both a stretching and a contraction,
and each horizontal segment is used as the standard of measurement for the other, there being no other unit of measurement available in the figure. Thus the angular effect is again doubled when viewed as a linear measurement effect.

We do not have as satisfactory an explanation for the weaker "strength" of the Zöllner illusion (Figs. 2 and 6). When corrections were plotted in steps of 0.005, the best illusions of parallelism were produced with |ε| = 0.015 and 0.02. (These two were difficult to distinguish, and the latter is shown in Fig. 6.) This suggests that the strength is half that of the other illusions considered, but we have not been able to identify a "halving" mechanism.

5. DISCUSSION OF HOFFMAN'S LIE-THEORETIC MODEL

This study began with an attempt to describe to persons not familiar with Lie group theory just what was being computed by Hoffman (1971). (Some familiarity with that paper is assumed for this section.) The Hoffman approach of identifying the distorted and distorting curves as orbits of Lie transformation groups acting in the plane, replacing the former by orthogonal trajectories, and forming a weighted linear combination of corresponding Lie derivatives to determine apparent curves as orbits of another group, is mathematically equivalent to the construction in Sections 2 and 3, with one important difference. It has already been noted by Copeland (1973) that Hoffman's association of Lie derivatives with families of curves is apparently done on an ad hoc basis and that equally plausible choices of Lie derivatives lead to qualitatively incorrect descriptions of the apparent curves. The essential point that distinguishes the present model from Hoffman's is the use of unit vectors for N and T, or equivalently, parameterization by arc length. Less important, but still worth noting as a difference, is Hoffman's appeal to subjective impressions of motion or direction for determining signs of coefficients in forming linear combinations. Not everyone will agree on subjective impressions, but, as we have seen, Brentano's Hypothesis uniquely determines the sign of ε in all cases.

This is not to say that there is anything incorrect or inappropriate about the basic underlying principles of Hoffman's model. Indeed, it appears to this author that Hoffman's theory may be precisely the more general theory of visual (and perhaps other) perception that Orbison predicted in 1939.5 One hopes that other workers sufficiently skilled in physiology, psychology, and mathematics will comment on the appropriateness of identifying perceptual constancies with mathematical invariances and neurons with infinitesimal generators. What this paper has attempted to show is that a normalized version of Hoffman's model serves quite adequately for descriptive purposes without any specific physiological assumptions about the underlying mechanisms of perception. The normalization is necessary because a correct Lie-theoretic model should not give qualitatively different results for different choices of coordinates.

5 Walker (1973, p. 483) states, "... a differential formulation of the theory might prove more generally applicable than the present formulation that depends on integral expressions involving straight line segments."
Some comment about the difference in details and interpretation arising from normalization are necessary. We begin with the examples that this paper and Hoffman's have in common.

Hoffman's analysis of the Hering illusion (Hoffman, 1971, pp. 181-183), formulated in the notation of Section 3, uses the same \( N \), but \( T = (x, y) \). Thus the distortion effect in forming \( N_a \) is not just a function of angle but also of distance from the origin. The choice of placing the origin at the center of the figure is both natural and computationally convenient, but it is not inevitable, and the shape of the apparent curve should not depend on it. Recall that we have demonstrated in Section 4 that the Ponzo illusion may be adequately corrected by treating it as mathematically equivalent to the Hering illusion. One of the consequences of this equivalence is that the natural center of the coordinate system is not only distinct from the center of vision, it is not even in the figure.

To compare the normalized and unnormalized models, we have plotted correction curves (arcs of circles) according to Hoffman (1971; Eq. (30)). The best such correction for the Hering illusion is shown in Fig. 11. The distinction between this figure and Fig. 8b is subtle, but we find a slight overcorrection in the center and undercorrection at the edges in Fig. 11. Adjustment of the "strength" parameter (not directly comparable to our \( \epsilon \)) in either direction enhances one or the other of these deficiencies.

![Fig. 11. Correction of the Hering illusion according to Hoffman (1971, Eq. (30)).](image)

It has already been noted in Section 2 that Hoffman's analysis of the Poggendorff illusion (Hoffman, 1971, pp. 178-180) specifically contradicts the Brentano Hypothesis. This results from his insistence that the apparent slope must equal the real slope. This is not required for mathematical consistency however, since the real slope does not appear in his differential equation, and the apparent slope enters only as an undetermined constant of integration. The essential problem here is precisely that the real slope does not appear, which follows from his interpretation of the transversal as an orbit of the

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\(^6\) See the Added Note at the end of this section.
dilation group (one of a family of rays through the origin) rather than as an orbit (or possibly parts of two different orbits) of a translation group (family of parallel lines). The latter approach, as we have seen, brings in both the real and apparent slopes explicitly.

The difficulty induced by using the dilation group becomes more apparent with the normalized model, or indeed, with any parameterization that differs from Hoffman's \([N = (-y, x)]\) other than by a constant factor: All such choices lead to solutions with nonzero curvature. Since the distortion effect occurs at exactly two points on the transversal, and not all along it as in the Hering and Orbison illusions, a valid apparent curve must have constant slope. [The Walker model (1973, p. 474) which predicts slight curvature near the intersection points is not relevant to this discussion, since it is based on a higher order effect not reflected in the differential equation model.]

Finally, there is the matter of magnitude of Poggendorff effect depending on the distance between the parallel lines. The strength of the illusion is obviously zero if the two lines coincide, and it is well-known that the effect is enhanced by moving the parallels further apart. Formula (4) does not show this effect directly, but the formula may be rewritten

\[
Ay = \frac{2mbe(m^2 + 1)^{1/2}}{e(m^2 + 1)^{1/2} + 1}.
\]

Since the first term in the denominator is small relative to 1, \(Ay\) is roughly proportional to \(e\). Following Hoffman (1971, p. 180), we believe the strength \(|e|\) to be (roughly) proportional to the horizontal separation, hence \(Ay\) is also. The assumption of constant \(|e|\) over changes in slope (not too large in magnitude) is appropriate for a given separation distance, but not in general. Thus it is merely fortuitous that the choice of 0.035, which works nicely for a wide range of illusions, is also about right for a conventionally drawn Poggendorff illusion. More will be said about this in the next section.

The difference of treatment for the Zöllner illusion is of no significance and will not be discussed.

If Hoffman had included the Orbison square illusion as an example, he would have used parameterizations (15) and (16), which correspond to his Lie derivatives \(L_a\) and \(L_0\), respectively. Without normalization, this leads to the differential equation

\[
\frac{dy}{dx} = \frac{ey}{1 + ex},
\]

which has solutions in the upper half plane given by

\[
y = k |x + (1/e)|.
\]

The only possible interpretation of (30) that leads to qualitatively correct apparent curves is that \(\text{sgn}(e) = \text{sgn}(x)\), so there are really two differential equations, and the apparent curve is a shallow "V" formed from parts of two different lines. We have seen how this conclusion about \(\text{sgn}(e)\) arises from Brentano's Hypothesis, but it is difficult to see how it can arise from subjective impressions about motion or direction when viewing the illusion. Furthermore, while the "V" shape is a reasonable description
of the apparent curve, some curvature appears to be necessary for an adequate correction curve, as indicated in Fig. 12, which shows correction curves according to (30) with $|e| = 0.015$. The illusion is overcorrected at the corners and undercorrected at the centers of the sides of the square.

Since the preparation of the original draft of this paper, the normalized model has been challenged by Hoffman (1975b, c, d) on several grounds that require some discussion.

The most substantial objection to the present model is that it rests squarely on Euclidean geometry in the plane of the visual illusion. It is therefore incapable of coping with anomalies of perceived distance (Oppel-Kundt illusion), variance of effect under rotation (preference for horizontal and vertical orientations), and other known non-Euclidean aspects of the geometry of the visual manifold. On this point there is no argument. We emphasized at the outset that the model is a first-order approximation, and its success depends on the fact that the aspect of visual perception under consideration is approximately Euclidean. The validity of the approximation is demonstrated by the accuracy with which illusions may be corrected. The other side of the question is whether Hoffman’s choice of “canonical” coordinates is more successful in dealing with non-Euclidean aspects of perception. The answer to this lies in the observation that Fig. 3 and Eq. (1) are equally applicable to Hoffman’s model if we drop the requirement that $\mathbf{N}$ and $\mathbf{T}$ be unit vectors. This leads to replacement of Eq. (14) by

$$\frac{dy}{dx} = \frac{g - eh}{f + ek},$$

(14')

where $f, g, h, k$ are associated with prescribed (“canonical”) parameterizations of the distorted and distorting curves. Alternatively, we can derive (14) from (14') (or vice versa) by replacing $e$ by

$$e' = e \left( \frac{f^2 + g^2}{h^2 + k^2} \right)^{1/2},$$

FIG. 12. Correction of the Orbison square illusion according to Hoffman’s model (1971).
that is, by allowing $c$ to vary in a prescribed way. (The Brentano Hypothesis does not require that $c$ should be constant.) Thus we see that Hoffman's apparent curves are solutions of similar differential equations, and his geometry is still Euclidean, in particular, rotation invariant. The test of whether it is more appropriate to normalize by arc length or to select canonical descriptions of the curves on a case-by-case basis is which produces the more accurate description of the apparent curves. We think our corrected illusions resolve that point in favor of normalization.

A second objection has to do with the introduction of initial conditions, which Hoffman (1975c) considers an unjustified additional hypothesis. The Lie-theoretic model is, of course, local in character, but the idea of solving the resulting differential equations for global solutions is Hoffman's (1971), not ours. He has since written that these "were never asserted as global solutions" (1975b), but that is not consistent with the discussion of the Hering illusion (especially Fig. 3b in Hoffman (1971)). Each such differential equation determines a family of curves, presumably including "the" apparent curve, although there is no reason why it should be unique with respect to position. The important feature of the curve is its shape, shared with all members of the family. Nevertheless, every curve in the family satisfies an initial condition at each of its points, and the selection principle used above for drawing corrections is a pragmatic one for getting the curve in the right part of the distortion pattern, the part giving rise to the differential equation in the first place.

A third point raised by Hoffman (1975c) is that the present model fails to explain Moiré effects, and in particular, the single-line Moiré pattern is a counterexample. It is indeed a counterexample to the modeling step described above in the paragraph containing Eq. (14). (That paragraph is unchanged from the first draft except for the addition of a footnote.) Using a continuous model for a discrete situation was not intended to be a universal principle, and the connectedness of the single-line Moiré clearly inhibits the type of processing that justifies the continuous model. However, the Brentano hypothesis is quite consistent with perceived Moiré effects, and work is under way on a discrete Brentano model that can be applied locally (i.e., at each point of intersection), both to Moiré patterns and to illusions for which the continuous model is also appropriate.

Finally, Hoffman's reliance on Lie transformation groups has led him to state (1975c), "[T]he patterns which may be involved in 'visual illusions of angle' are limited either to orbits involving the psychological constancies ... or ones generated by linear combinations of the associated Lie derivatives. Thus a spiral, representing a combination of size and rotation constancies, may be involved in such a visual illusion, but a sinusoid, for example cannot.... [T]he Poggendorff illusion is possible with oblique parallel lines ... but not sinusoids...." We have disposed of this by creating (and correcting, via the model described herein) three different illusions utilizing sinusoids, both as distorted and distorting curves, one of which shows a Poggendorff-type separation (Smith, 1977).

Added note. Since the original preparation of this section, the work of Day and Dickinson (1976) on the components of the Poggendorff illusion has appeared. Their work supports Hoffman's position that real and apparent slopes are equal, but it also indicates the extent to which a rotationally invariant model (ours or Hoffman's) is at
best an approximation, since they find that almost 35% of the displacement effect is attributable to the horizontal-vertical (HV) illusion of extent. The remainder they attribute to obtuse angle (OA) and longitudinal-transverse (LT) illusions of extent (40 and 25%, respectively), rather than to misperception of angles. This interpretation would seem to classify the Poggendorff illusion (and by implication the Müller-Lyer illusion, as well) as illusions of extent rather than of angle. However, it seems to have no bearing on differential equation model (14) and the illusions to which it applies, since real and apparent slopes are clearly different in these cases.

6. Implicit Assumptions and Possible Extensions

Thus far we have stated explicitly only two assumptions on which our descriptive model is based: the statement of Brentano's Hypothesis and the working assumption for implementing it that the strength parameter $|\epsilon|$ is constant for a given illusion. By informal experimentation, $|\epsilon|$ has been found to be quite consistent over a range of classical illusions, using ability to "correct" the illusion as a test. This consistency also holds up for other illusions not shown here, such as distortions of lines by families of hyperbolas or sinusoids (Smith, 1977).

Nevertheless, $|\epsilon|$ is not really constant, as the well-known dependence of the Poggendorff illusion on separation of the parallels shows. Nothing in Brentano's Hypothesis requires that $|\epsilon|$ should be constant, of course, since the differential equations for apparent or corrected curves can be formulated just as well with a strength that is dependent on position.

Under what circumstances can the strength be expected to be constant? Alternatively, what factors affect the strength of an illusion (and therefore have been assumed to be held constant in the previous discussion)? In order to see the illusions and the "corrections" properly, they must be relatively central in the field of view and not too large. Contrast between figure and ground, linewidth, set, and context all have some bearing. The relative density of the distortion pattern is particularly important: With either too few lines or too many, the illusion disappears. However, it does not have to be completely uniform, as the Hering illusion shows. Indeed, with crossing points evenly spaced on the parallels the illusion is considerably weaker than with equal central angles as shown in Fig. 8. Orientation in the field of view also affects the strength of the illusion, as one may verify by slowly rotating the Hering figure.

Some tacit assumptions have also been made about the observers, i.e., the readers of this paper. For example, it is known that cultural differences affect the strength of illusions (Segall, Campbell, & Herskovits, 1963). In particular, we have assumed that the set of readers does not include small children or members of primitive societies. It is also known (Teuber, 1960, p. 1656) that repeated exposure to the same illusion in the same orientation weakens or even destroys the effect, particularly when the repetitions are massed in a short time period. (The author has experienced this during the preparation of this paper with regard to the Hering illusion in the orientation shown. This happened to be the earliest illusion studied in this project.)
All of the factors mentioned, and others, such as the additivity effects noted in Section 4 with regard to the Müller–Lyer illusion, have been extensively studied, and it would not be difficult to determine their relationship to the strength parameter under controlled conditions. The results of such experiments would probably contribute to the solution of a much more interesting problem for those with appropriate skills and backgrounds, namely, determination of the neurophysiological basis of the strength parameter, if, as we think to be the case, there is a systematic and quantitative validity to Brentano’s Hypothesis. It may be too much to expect that there is a single neurophysiological entity that corresponds exactly to “strength,” but it may be helpful to focus attention on discovery of the components of this (as yet, purely mathematical) concept.

It has been suggested by Walker (1973, 1975) that the distinction between distorted and distorting curves is an artificial one and that the subtle bending effects in the distortion pattern should also be corrected. This is consistent with the Brentano Hypothesis, if not required by it. Symmetric treatment of all angles will be a feature of the discrete model mentioned in the previous section, which will eliminate the need for artificially separating the families of curves.

ACKNOWLEDGMENTS

The author’s interest in this problem was stimulated by the work of Mr. Howard Shapiro while an undergraduate at Duke University. Thanks are due to Professors Charles White (Psychology) and Murray Cantor (Mathematics) of Duke for their helpful conversations on this subject. Special thanks go to Professor William Hoffman for his comments (Hoffman, 1975a) on the earliest draft of this work, which helped greatly to shape and focus the author’s thinking on the subject. While we still disagree on a number of fundamental points, his commentary (Hoffman, 1975b, 1975c, 1975d) has continued to be useful.

The illustrations were prepared on a COMPLTDP-7 plotter at Duke University Computation Center, the staff of which has been very helpful. Computations were done on an IBM 370/165 at Triangle Universities Computation Center. Mr. Larry Parsley and Ms. Karen Anderson assisted with the programming work, and their services were compensated by the Duke Undergraduate Research Assistantship Program.

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\* In particular, \( \epsilon \) should be compared to the single parameter in Walker’s model (1973), which provides a possible explanation for Brentano’s Hypothesis.


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**RECEIVED:** February 22, 1977