Deuteranomaly Studied with Four Perceptual Criteria

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Received 17 August 1994; in revised form 2 February 1995; in final form 14 December 1995

In the theoretical section of the present paper, we develop our view of the roles played by the perceptual criteria: indistinguishably equal, neither blue nor yellow, neither green nor red, and heterochromatically equally bright. These criteria constitute a vectorial opponent-colour space, a concept used throughout the paper. Within this framework, two new theorems on psychophysical opponent-colour channels are stated. In the experimental section, the perceptual criteria are applied to the colour vision of an (extreme) deuteranomalous male. A quantitative perceptual description of his deuteranomaly is developed, the main constituents of which are measured deuteranomalous colour matching functions, the deuteranomalous opponent-colour functions derived from them and taking quantitative account of the Abney effect. Copyright © 1996 Elsevier Science Ltd.

Deuteranomaly Chrominance Luminance Perceptual criteria Opponent-colour trilateral (Equations) homogeneous in tristimulus values

INTRODUCTION

The usual characterization of anomalous trichromacy by means of an anomaloscope (Nagel, 1907; Heinsius, 1973), based as it is on the colour match, i.e., the perceptual criterion “indistinguishably equal”, does not capture most of the perceptual qualities of anomalous trichromatic vision. We find that the application of the three following perceptual criteria: “neither blue nor yellow”, “neither green nor red”, and “heterochromatically equally bright” leads to a perceptive description and lays the foundation of a three-dimensional opponent-colour theory (Scheibner, 1987; Scheibner & Wolf, 1984, 1985, 1985/86).

The initial, theoretical section of the present paper shows how to establish a linear transformation from any linear colour space into a linear opponent-colour space. These derivations culminate in two new theorems on opponent-colour transfer channels. In the second section, the experimental results obtained with a deuteranomalous observer are presented. Applying the theoretical results to these data, we are able to derive a mapping from an instrumental colour space to a deuteranomalous opponent-colour space.

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DETERMINING LINEAR TRANSFORMATIONS OF COLOUR SPACES BY MEANS OF PERCEPTUAL CRITERIA

We assume an initial three-dimensional instrumental colour space $V_{in}$ that is technically determined by the optical output of a visual tristimulus colorimeter. A colour $C$ may be represented by a vectorial colour equation of the form

\[ C := B \cdot B + G \cdot G + R \cdot R \in V_{in} \]  

where $(B, G, R)$ are tristimulus values, $B$, $G$, $R$ are reference stimuli, so-called primaries, and $V$ stands for a three-dimensional vector space.

A well-known example of such an instrumental primary system is that of Wright (1946), in which the primaries are defined by monochromatic radiations of wavelengths 460 nm (blue, $B$), 530 nm (green, $G$) and 650 nm (red, $R$). The symbol $: = \in V_{in}$ in equation (1) means "by definition equal to". However, equation (1) may also be interpreted to mean "looks the same as", in which case equation (1) describes a colour match and the symbol connecting the two members of the equation may be replaced by $\equiv$, which is then to mean an equivalence relation called a "metameric relation" (Scheibner, 1966, 1968, 1969).

We wish to show that the colour match fails to establish a linear transformation from an instrumental colour space $V_{in}$ onto an opponent-colour space $V_{opp}$. The three additional perceptual criteria mentioned in the
introduction, however, are demonstrated to be sufficient for establishing such a transformation.

Let a colour stimulus be specified by equation (1) within the instrumental colour space, as well as within an opponent-colour space by the following vectorial equation:

\[ C := MM + SS + LL \epsilon v_{\text{Opp}} \]  

where \((M, S, L)\) are opponent tristimulus values and \(M, S, L\) are opponent primaries (Scheibner, 1987; Scheibner & Wolf, 1985; Scheibner & Lochner, 1991; Lochner & Scheibner, 1991; Knottenberg & Scheibner, 1993). The connection between the two linear spaces \(v_{\text{ln}}\) and \(v_{\text{OPP}}\) may be written (e.g. Reichardt, 1957; Kostrikin & Manin, 1989)

\[(M, S, L) = (B, G, R)A, \quad \text{where } A \text{ (or its inverse } A^{-1}) \text{ is a regular three-dimensional matrix.} \]

If we write out equation (4), we have three vectorial equations, each of which is of the same type as equation (1), their left sides reading \(M, S, L\), respectively, instead of \(C\), and with the equality sign = instead of :=.

As stated with equation (1), we can interpret these equations also as colour matches:

\[ M = BMB + GMG + RMR \]
\[ S = BsS + GsG + RsR \]
\[ L = BLS + GLG + RLR \]

Operationally, colour matches of this kind can be performed in a bipartite field of a visual tristimulus calorimeter by presenting, say, the radiation stimulus described by \(L\) in one half of the field and matching it in the other half to the additive superposition of the three instrumental reference stimuli described by \(B, G, R\).

Figure 1 shows the triangle of the three instrumental reference stimuli \(B, G, R\) within the chromaticity chart, along with two additional triples of colours \(M', S', L'\) and \(M, S, L\). Only the colours \(M', S', L'\) lie within the range of real colours and, hence, can be reproduced by additive mixing of the reference stimuli \((B, G, R)\), whereas the colours \(M, S, L\) cannot, because \(M\) and \(S\) are imaginary, i.e., non-physical, and cannot be used in a colour match. The triangle \((M, S, L)\) of Fig. 1 represents, however, the typical constellation of an opponent-colour system (Scheibner & Wolf, 1984). Hence, it is not possible to determine the matrix \(A\) (or \(A^{-1}\)) via equations (4) or (5) through colour matches if the aim is an opponent-colour system \((M, S, L)\).

Instead of using equations (4) or (5) in applying the colour match, we now use equation (3) in applying the three perceptual criteria mentioned earlier to determine the matrix \(A\). Written out, equation (3) consists of the three scalar equations

\[ M = \beta_M B + \gamma_M G + \rho_M R \]
\[ S = \beta_S B + \gamma_S G + \rho_S R \]
\[ L = \beta_L B + \gamma_L G + \rho_L R. \]

The three perceptual criteria may be reformulated in the following way: (a) no blue–yellow chroma is described by the colorimetric quantity \(M\); (b) no green–red chroma is described by the colorimetric quantity \(S\); (c) no brightness is described by the colorimetric quantity \(L\). The statement (c) on brightness will be additionally illuminated by equation (9a) at the end of the experimental section. For colours that fulfil these criteria, i.e., do not exhibit the perceptual qualities described by \(M, S, L\), respectively, the left members of equations (6) take on the value zero.† They then describe planes containing the origin of the vectorial colour space, i.e., they describe two-dimensional subspaces of the vectorial colour space. Their intersections with the chromaticity chart form straight lines designated by \(M = 0, S = 0, L = 0\) in Fig. 1.

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*The notation \(M\) goes back to Luther (1927) meaning chromatic moment. Thus, \(M\) may mean first chromatic moment, \(S\) second chromatic moment. \(L\) stems from luminance.

†In other words, the opponent tristimulus values \(M, S, L\) change sign.
The equations (6) have become homogeneous linear forms that describe planes within the colour space as well as straight lines within the chromaticity chart. The tristimulus values B, G, R may be considered as general projective homogeneous coordinates (Maxwell, 1963). Such a linear form reads:

\[ 0 = \beta_M M + \gamma_M G + \rho_M R \]  
\[ (6a) \]

This type of equation is, in contrast to equation (6), experimentally accessible through a perceptual criterion of the kind mentioned.

Thus, the task of establishing a transformation into an opponent-colour space according to equations (6) may be divided into the following steps:

1. Measure chromaticity loci in the form of straight lines through each of the perceptual criteria (see section on experimental methods). The straight lines are usually expressed in chromaticity coordinates, here \( r \) and \( g \), in the form

\[ g = h + mr \]  
\[ (7) \]

where \( h \) and \( m \) are constants for a given line.

2. Rewrite the straight lines [equation (7)] as linear forms [equation (6a)] by means of the relations

\[ b = \frac{B}{B + G + R}, \quad g = \frac{G}{B + G + R}, \quad r = \frac{R}{B + G + R} \]  
\[ (8) \]

3. Return from the plane equations of type (6a) to the mapping equations of type (6).

The equations of type (6a), which we have considered as describing planes, may also be viewed as mapping equations, namely two-dimensional mapping kernels (e.g. Reichardt, 1957; Kostrikin & Manin, 1989). These kernels are mapped to zero. What the perceptual criteria do is to single out these two-dimensional kernels. Step 3, then, implies generalizing from the kernel's mapping (6a) to the full mapping (6). The quantities \( M, S, L \) appearing in step (3) are opponent tristimulus values. We call \( M \) "blue-yellow chrominance", \( S \) "green-red chrominance", and \( L \) "luminance". As all real colours are located only on one side of the straight line \( L = 0 \), there are, in contrast to \( M \) and \( S \), no psychophysical sign changes for \( L \).

Historically, the third line of equation (6) has been known as Abney's equation (Abney & Festing, 1886); the coefficients have been called luminance coefficients. The corresponding subspace, i.e., plane, was named "alychna" by Schrödinger (1925). Analogously, the coefficients in the first and second lines of equation (6) may be called chrominance coefficients, the set \( (\beta_M, \gamma_M, \rho_M) \) being blue-yellow chrominance coefficients, the set \( (\beta_S, \gamma_S, \rho_S) \) green-red chrominance coefficients.

### TWO THEOREMS CONCERNING OPPONENT-COLOUR TRANSFER CHANNELS, AND THEIR TRANSFORMATIONAL PROPERTIES

While the concept of opponent-colour channels is well established in the theory of colour vision (see e.g. Blum, 1991; Brannan, 1992), their morphological and/or physiological specification is still rather uncertain. The foregoing results suggest the first of our theorems, by which the concept of channels receives a simple and, at the same time, precise meaning:

The three psychophysical opponent-colour channels of trichromatic vision are defined by three specific two-dimensional subspaces of a linear colour space.

Proof: The three subspaces postulated are given by homogeneous linear forms and may be expressed by

\[ \beta_M B + \gamma_M G + \rho_M R = \langle \Gamma_M, C \rangle = 0 \]
\[ \beta_S B + \gamma_S G + \rho_S R = \langle \Gamma_S, C \rangle = 0 \]  
\[ (9) \]

where \( C \) stands for the triple \( (B, G, R) \) of tristimulus values, capital gamma \( \Gamma_i \) (\( i = M, S, L \)) for the triples \( (\beta_i, \gamma_i, \rho_i) \) of chrominance and luminance coefficients, and the angle brackets \( \langle < \rangle \) denote a scalar product. Equation (9) arises from setting the left sides of equations (6) to zero, in agreement with the three reformulated perceptual criteria. However, since equations (6) define the three colorimetric quantities \( M, S, L \) that correlate with the perceptual opponent-colour qualities, equations (6) also embody the transfer channels.

In order to formulate our second theorem, we observe that in equation (9) the triples \( (B, G, R) \) are members of the given vector colour space, and that the triples \( (\beta_i, \gamma_i, \rho_i) \) are members of the dual vector space associated with the colour space (Scheibner, 1993). The linear forms in equation (9) are also scalar products.

We pose the problem of how to transform chrominance and luminance coefficients, if the underlying tristimulus values are transformed. The answer: The matrix that describes the transformation of chrominance and luminance coefficients is derived by transposing and inverting the matrix that describes the transformation of tristimulus values (e.g., Reichardt, 1957; Kostrikin & Manin, 1989). Thus, the second theorem may be stated as follows:

If a vectorial colour space is subjected to a linear transformation, the chrominance and luminance coefficients governing the colour-opponent channels associated with the given colour space are transformed contragradiently to the tristimulus values.

An explanatory, numerical example has been presented elsewhere (Scheibner, 1993).

### EXPERIMENTAL METHODS

The experiments were performed by an extreme deuteranomalous observer, the second author. His matching range in Nagel's anomaloscope was \( 1 < AQ < \infty \), where \( AQ \) is the anomalous quotient (see e.g. Heinsius, 1973; Pokorny et al., 1979). We used a visual tristimulus colorimeter built according to the Guild–Bechstein model (Beck & Richter, 1958). The diameter of the bipartite visual field was 2 deg and the luminance of the visual field was between 60 and 110 cd/m\(^2\) with a 1.6 mm diameter exit pupil yielding...
between 120 and 220 troland. The luminance was measured by means of Bechstein's visual luminance meter (manufactured c.1960 by Schmidt & Haensch, Berlin). The instrumental primaries (reference stimuli) of the colorimeter were realized by approximately monochromatic radiations of wavelengths 460 nm (B), 530 nm (G) and 650 nm (R). The physical, real nature of these primaries offers the advantage that any type of observer (normal as well as various anomalous trichromats) can be characterized. This is so because results of colour matches can always, without any pre-conditions, be plotted with reference to the radiations used as matching primaries. The various measured objects must, of course, be designated as such. In the following figures (Figs 2–4), therefore, the colour matches of our deuteranomalous observer are plotted with reference to the instrumental primaries used, together with a normal trichromat’s spectral data measured under the same conditions, namely by Wright (1946). To stress the difference, plotting together normal and anomalous trichromats can always, without any pre-conditions, be plotted with reference to the radiations used as matching primaries. The various measured objects must, of course, be designated as such. In the following figures (Figs 2–4), therefore, the colour matches of our deuteranomalous observer are plotted with reference to the instrumental primaries used, together with a normal trichromat’s spectral data measured under the same conditions, namely by Wright (1946). To stress the difference, plotting together normal and anomalous trichromats with a common reference to imaginary primaries, e.g., those of the system CIE 1931, would not be correct.

First, the colour matching functions of the subject Th. K. were measured under the condition of minimal saturation (Maxwell, 1880; Crawford, 1965; Bruckwilder & Scheibner, 1988/89). The spectral radiations to be matched by superposition of the three primaries were isolated by means of interference filters. These radiations were also measured radiometrically in order to calculate the colour matching functions. We used the Tektronix Radiometer J 16, calibrated in mW/m².

Next, the perceptual criterion “neither blue nor yellow” was performed: in the left half of the visual field, binary colour mixtures were varied and perceptually judged until they obeyed the criterion while the right half of the field was black. Those special binary mixtures were then colorimetrically specified by matching them to a mixture of the three instrumental primaries in the right half of the bipartite visual field. The resulting chromaticity loci were plotted in the chromaticity chart. The analogous procedure was performed with the perceptual criterion “neither green nor red”.

In applying the perceptual criterion “heterochromatically equally bright” (Scheibner, 1993), pairs of colours \(C_0, C_j\), \(j = 1–7\), were equated in luminance. The method of minimally distinct border was used (Boynton, 1978). The members of the pairs \(C_0, C_j\) were then matched individually to a superposition of the three instrumental reference stimuli and thus colorimetrically specified within the (B, G, R) system. The vectorial difference between the members of each pair

\[
(B_0 - B_j, G_0 - G_j, R_0 - R_j)
\]

(9a)

was then calculated. Such difference vectors are by definition luminance-free. It is for such difference vectors that the reformulated achromatic criterion “no luminance” holds; their chromaticity loci were plotted in the chromaticity chart.
FIGURE 3. Chromaticity chart pertaining to the instrumental \((B, G, R)\)-system. The solid symbols are chromaticity loci of colours that obey the perceptual criterion "neither blue nor yellow" as judged by the deuteranomalous observer Th.K. The global linear approximation (long dashed line) is based on a linear regression equation fitted to all the data. The piecewise linear approximations are shown as the continuous lines together with their small dashed extensions. They indicate linear equations fitted to portions of the data. Note that here and in Fig. 4, the spectral locus and the wavelengths written beside it pertain to the normal trichromat, not to our deuteranomalous trichromat (see text).

FIGURE 4. Similar to Fig. 3, but indicating the chromaticity loci that obey the perceptual criterion "neither green nor red".
RESULTS

Figure 2 shows the colour matching functions \( \bar{b}(\lambda), \bar{g}(\lambda), \bar{r}(\lambda) \). The curves designated with symbols are the measured deuteranomalous ones. For comparison, the dashed curves represent the standard colour matching functions for the normal trichromat according to Stiles–Burch–Estévez (Estévez, 1982). Being an extreme deuteranomalous observer, the subject exhibited wide matching ranges in the red–green region. In an attempt to minimize these ranges, he approached the matching points by either solely “coming from too red” (circles with a central dot) or solely “coming from too green” (circles with a vertical diameter). The solid circles are their means. The purpose of these deuteranomalous colour matching functions \( \bar{b}(\lambda), \bar{g}(\lambda), \bar{r}(\lambda) \) was to transform them into deuteranomalous opponent spectral functions \( \tilde{m}(\lambda), \tilde{s}(\lambda), \tilde{t}(\lambda) \).

Figure 3 shows a \((r, g)\)-chromaticity chart pertaining to the instrumental colorimetric system \((B, G, R)\). The dots denote chromaticity loci of binary mixtures that obeyed the criterion “neither blue nor yellow”. Various straight lines are global or partial straight line approximations of these points. The straight line between 474 and 560 nm is an example of a line of varied binary colour mixtures presented and judged in the left side of the bipartite visual field. Figure 4 shows the analogous result for the perceptual criterion “neither green nor red”.

Figure 5 shows the results of the heterochromatic luminance matches. The pairs of colours equated in luminance are denoted by \((C_0, C_j)\), \(j = 1–7\). The chromaticity loci of the luminance-free difference vectors are denoted by \(D_j, j = 1–7\). They are well approximated by straight lines, the so-called trace of the alychné. If the line segments connecting \(C_0\) and \(C_i\) are considered as binary mixture lines, the line segments connecting \(C_i\) and \(D_i\) are exterior parts of these binary mixture lines, i.e., line segments on which a binary mixture has one negative component.

The lines approximating the measured chromaticity loci in Figs 3–5 are characterized by equations expressed in chromaticity coordinates \((r, g)\) and, moreover, by equations homogeneous in tristimulus values \((B, G, R)\).

For the criterion “neither blue nor yellow” \((M, \text{Fig. 3})\), the homogeneous equations are

\[
0.9923B - 0.1975G - 0.069R = 0, \quad (10)
\]

for the global approximation, and

\[
0.6427B - 0.0994G - 0.2402R = 0, \quad (11)
\]

\[
0.9833B - 0.4439G + 0.0155R = 0 \quad (12)
\]

for the piecewise linear approximation, the upper and lower parts of which are joined at \(g = 0.44\). In order to make the spectral function \(\tilde{m}((\lambda))\) continuous, equation (12) is also used in a different normalization:

\[
0.5870B - 0.2650G + 0.0092R = 0. \quad (13)
\]

The criterion “neither red nor green” \((S, \text{Fig. 4})\) leads to the global approximation

\[
0.2782B - 1.1643G + 1.2261R = 0 \quad (14)
\]
and the piecewise linear approximations
\[ 0.3861B - 1.0346G + 1.0542R = 0, \]  
\[ 0.0462B - 0.3349G + 1.8901R = 0 \]
with upper and lower parts joined at \( g = 0.28 \).

Finally, the heterochromatic luminance matches (L, Fig. 5) result only in one approximation
\[ 0.0473B + 0.9157G + 0.3793R = 0. \]

**DERIVATION OF AN OPPONENT-COLOUR SPACE (M, S, L) AND SPECTRAL OPPONENT FUNCTIONS \( \hat{m}(\lambda), \hat{s}(\lambda), \hat{t}(\lambda) \)**

In view of equations (6) and the evaluation steps subsequently delineated, we gather the homogeneous linear forms in Figs 3–5 and supplement them to the transformation equation (6). Using the global approximations in Figs 3 and 4 (long dashes), given by equations (10), (14) and (17), we obtain

\[
\begin{align*}
M &= 0.9923B - 0.1975G - 0.0690R \\
S &= 0.2782B - 1.1643G + 1.2261R \\
L &= 0.0473B + 0.9157G + 0.3793R
\end{align*}
\]

These equations effect the mapping

\[
\begin{bmatrix}
B \\
G \\
R
\end{bmatrix} \rightarrow \begin{bmatrix}
M \\
S \\
L
\end{bmatrix}
\]

of the colour vectors in the instrumental colour space into colour vectors of a deuteranomalous opponent-colour space.

The three colour matching functions

\[
\begin{bmatrix}
\hat{b}(\lambda) \\
\hat{g}(\lambda) \\
\hat{r}(\lambda)
\end{bmatrix}
\]

are simply tristimulus values of the equal energy spectrum. Hence, the deuteranomalous colour matching functions of Fig. 2 may be transformed by means of equation (18), resulting in the colour-opponent functions shown in Fig. 6.

In Figs 3 and 4 the deviations from a straight line were captured by piecewise linearization. These segments of straight lines give rise to the piecewise transformations shown in Fig. 7; they correspond to the three wavelength intervals \( \lambda \leqslant 491 \text{ nm}, 491 \text{ nm} \leqslant \lambda \leqslant 575 \text{ nm}, \) and \( 575 \text{ nm} \leqslant \lambda \). The precise wavelengths were determined by trial and error to achieve continuous curves. The valid branches of these opponent spectral functions are drawn in full lines, the invalid continuations are drawn as dashed and dotted curve branches. The valid branches in Fig. 7 result from the continuous line segments in Figs 3 and 4 and the zero-luminance trace in Fig. 5; the invalid branches in Fig. 7 result from the line segments drawn in short dashes in Figs 3 and 4.

For the wavelength interval \( \lambda \leqslant 491 \text{ nm} \) the deuteranomalous opponent spectral functions read (Fig. 7), using equations (12), (16) and (17):

\[
\begin{align*}
\hat{m}(\lambda) &= 0.9833\hat{b}(\lambda) - 0.4439\hat{g}(\lambda) + 0.0155\hat{r}(\lambda) \\
\hat{s}(\lambda) &= 0.0462\hat{b}(\lambda) - 0.3349\hat{g}(\lambda) + 1.8901\hat{r}(\lambda) \\
\hat{t}(\lambda) &= 0.0473\hat{b}(\lambda) + 0.9157\hat{g}(\lambda) + 0.3793\hat{r}(\lambda)
\end{align*}
\]

**DISCUSSION OF THE EXPERIMENTAL PART**

The measured deuteranomalous colour matching functions of Fig. 2 (curves provided with symbols) show the typical spectral shape exhibiting negative branches and zero crossings. Because the zero crossings are fixed by the wavelengths of the instrumental primaries, displacements along the wavelength axis do not emerge so clearly as with so-called fundamental spectral curves (or psychophysical cone fundamentals). Tabulated values for fundamental spectral curves (including anomalous trichromatic ones) have been published by DeMarco et al. (1992). Unfortunately, these authors do not report transformation equations connecting an instrumental reference system with spectral fundamental functions. Thus far, we have not been able to derive such equations in order to transform our measured deuteranomalous colour matching functions and compare them with the deuteranomalous psychophysical cone fundamentals of DeMarco et al. (1992). Nevertheless, the main branches of the measured curves \( \hat{g}(\lambda) \) and \( \hat{r}(\lambda) \) show a distinct shift towards longer wavelengths compared with the normal trichromatic observer that is shown in dashed curves in Fig. 2 following Estévez (1982).

The single pigment shift hypothesis of deuteranomaly claims that only the “green” (MWS*) cone pigment—represented in Fig. 2 by the function \( \hat{g}(\lambda) \)—should exhibit a shift towards longer wavelengths (e.g., Pokorny & Smith, 1975). This property is also incorporated in the anomalous psychophysical cone fundamentals of DeMarco et al. (1992). Our results clearly show this

\*MWS = Middle Wavelength Sensitive.
Opponent spectral functions
(no piecewise linearisations)

FIGURE 6. Deuteranomalous opponent spectral functions derived by means of the global linearization of Figs 3 and 4, and the straight line connecting the dots D1–D7 of Fig. 5. The equations shown are the same as equations (18).

property, but they do not exclude a shift also of a second cone pigment, namely the long wavelength pigment, towards longer wavelengths.

The results of Fig. 7 show that deuteranomalous colour opponency is similar to normal trichromatic opponency. That the three curves \( \bar{m}(\lambda), \bar{s}(\lambda) \) and \( \bar{l}(\lambda) \) can be normalized independently follows from the homogeneous equations (9) or (10)–(17). We have yet to find a satisfactory way of normalization so that the green–red mechanism \( \bar{s}(\lambda) \) may appear weakened compared to the blue–yellow mechanism \( \bar{m}(\lambda) \).

In addition to the global linearization of the zero chrominance traces of Figs 3 and 4, piecewise linearizations are calculated and plotted. They express an Abney effect. The Abney effect (Westphal, 1909; Abney, 1910; Burns et al., 1984; Kurtenbach et al., 1984; Kremer, 1992; Knottenberg & Scheibner, 1993) states that desaturation of spectral colours changes both chromatic saturation and hue. For the green–red mechanism shown in Fig. 4, this effect is rather pronounced. It is less pronounced for the blue–yellow mechanism (Fig. 3). These findings are in general agreement with the reports of Burns et al. (1984) and Kurtenbach et al. (1984). Thus, our deuteranomalous trichromat shows an Abney effect that is similar to that of a normal trichromat.

The piecewise linearizations in Figs 3 and 4 result in three intervals of the wavelength axis in Fig. 7. The respective curve branches are denoted by various solid symbols, and the pertaining equations are the equations (19)–(21). The zero crossings of the opponent spectral functions (Figs 6 and 7) denote the sign changes of chrominances and characterize, therefore, the wavelengths of the unique (or primary) hues. Within the chromaticity chart, the sign changes for chrominances occur in the region of real colours (Figs 3 and 4), the sign changes for the luminance occur exclusively in the region of imaginary (non-physical) colours (Fig. 5). The three traces of sign changes in the form of straight lines form the opponent-colour trilateral. In Fig. 1, the trilateral is schematically demonstrated: It consists of the traces \( M = 0 \) and \( S = 0 \), i.e., Hering’s “axis cross”, and the trace \( L = 0 \), i.e., Schrödinger’s “alychne trace” (Schrödinger, 1925). Due to the Abney effect, the sign change of the curve \( \bar{s}(\lambda) \) below 500 nm in Fig. 7, i.e., unique blue, is displaced towards shorter wavelengths, which can also be seen in Fig. 4. Unique green is shifted towards longer wavelengths, which can also be seen in Fig. 3. Unique yellow is not influenced by the Abney effect and has a wavelength between 580 and 590 nm, which cannot be seen in Fig. 4, because the wavelength numbers written beside the normal trichromat’s spectral locus are those of the normal, not the deuteranomalous trichromat. These results are in accordance with the reports of Kurtenbach et al. (1984). One feature of our results is, however, that they were obtained without fixing any arbitrary white. This was possible because we confined ourselves to the unique hues (including red), and the observer judged binary colour mixture variations according to a perceptual criterion with consecutive colour matching of the chosen mixture.
The deviations of the chromaticity loci from straight lines in Figs 3 and 4 suggest non-linear processes, probably somewhere in postreceptoral visual pathways. We are not aware of a recent, complete quantitative formulation of such non-linearities or of a formulation that takes desaturated colours or deuteranomalous observers into account. Previous attempts (Larimer et al., 1974, 1975; Elzinga & de Weert, 1984) model the non-linearities by means of retinal cone excitations following a power law. Such a model may allow for large variations in stimulus intensity (Bezold–Brücke effect). In our experiments, however, the luminance level was maintained between 60 and 110 cd/m². Burns et al. (1984) discussed alternatives to power functions, but they did not report on a complete quantitative description of the Abney effect.

For any given colour stimulus, the deuteranomalous opponent tristimulus values M, S, L characterizing colour attributes can be calculated according to the following rules

\[
\begin{align*}
M &= C_m \int \tilde{m}(\lambda)\phi_\lambda d\lambda \\
S &= C_s \int \tilde{m}(\lambda)\phi_\lambda d\lambda \\
L &= C_l \int \tilde{l}(\lambda)\phi_\lambda d\lambda,
\end{align*}
\]

where \(\phi_\lambda\) is the spectral radiation distribution of the colour stimulus and \(C_m, C_s, C_l\) are normalizing constants. These equations reflect a procedure well known in colorimetry (cf. e.g., Berger–Schunn, 1991). In the case of the present piecewise linear approximations, the integration has to be done within the interval limits given with equations (19)–(21) (cf. Fig. 7). The colorimetric attribute M, the blue–yellow chrominance, correlates perceptually with blueness (\(M > 0\)) and yellowness (\(M < 0\)); similarly S, the green–red chrominance, correlates perceptually with greenness (\(S < 0\)) and redness (\(S > 0\)), and L, luminance, correlates perceptually with brightness (\(L > 0\)).

In summary, the deuteranomalous colour matching functions \(\hat{b}(\lambda), \hat{g}(\lambda), \hat{P}(\lambda)\), in combination with equations (19)–(22), provide a nearly complete, quantitative perceptual description of deuteranomaly that includes the Abney effect.

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Acknowledgements—We thank Sinclair Cleveland and an anonymous reviewer for critical comments and helpful suggestions, Claudia Wittrock, Petra Schwarz and Erika Scheibner for technical assistance.