An integrated wave-effects model for an underwater explosion bubble

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(Received 28 June 2001; revised 31 December 2001; accepted 14 January 2002)

A model for a moderately deep underwater explosion bubble is developed that integrates the shock wave and oscillation phases of the motion. A hyperacoustic relationship is formulated that relates bubble volume acceleration to far-field pressure profile during the shock-wave phase, thereby providing initial conditions for the subsequent oscillation phase. For the latter, equations for bubble-surface response are derived that include wave effects in both the external liquid and the internal gas. The equations are then specialized to the case of a spherical bubble, and bubble-surface displacement histories are calculated for dilational and translational motion. Agreement between these histories and experimental data is found to be substantially better than that produced by previous models. © 2002 Acoustical Society of America. [DOI: 10.1121/1.1458590]

PACS numbers: 43.25.Yw [ADP]

I. INTRODUCTION

The early work on bubble dynamics focused on underwater explosions [Lamb (1923); Herring (1941); Kirkwood and Bethe (1942); Penney and Price (1942); Taylor (1942); Shiffman and Friedman (1943); Ward (1943)]; this work is well covered in Cole (1948). Since the 1940s, the field has expanded to encompass such areas as medical ultrasonics, sonochemistry, and oceanography [Leighton (1994)]. In recent years, there has been a resurgence of interest in underwater explosion bubbles, driven by the increasing use of computational methods to predict the response of marine structures to underwater explosions [see, e.g., Atkatsh et al. (1983); DeRuntz (1989); Schittke et al. (1989)].

A. Bubble phenomena

The sequence of events in the motion of an underwater explosion bubble is depicted in Fig. 1 [Snay (1956)], where the snapshots are continuously offset to the right for clarity. When the detonation wave in a spherical charge reaches the charge’s wet surface, the highly compressed gas in the small bubble propels the surrounding liquid radially outward, generating an outward-propagating shock wave. The shock-wave pressure profile is step-exponential in nature; 10 m from a 100-kg charge of TNT, the maximum pressure is approximately 20 MPa (3000 psi) and the time constant is about 1/2 ms. Unlike a cavitation bubble generated by an acoustic transducer or a ship’s propeller, the bubble produced by an explosive charge is dirty, containing a mixture of explosion products.

As the dilating bubble continues to expand, the pressure inside decreases until it is below the ambient (hydrostatic plus atmospheric) pressure, and soon (at about 100 ms for a 100-kg charge at a depth of 100 m) the bubble reaches its maximum radius (of 3 m or so for the 100-kg/100-m case) and begins to contract. The contraction proceeds until the bubble collapses on itself (at about 200 ms for the 100-kg/100-m case) and then rebounds; the duration of the collapse–rebound segment (roughly 10 ms for the 100-kg/100-m case) is much shorter than the time to first collapse/rebound. During this segment, a second outward-propagating wave is generated, which carries nearly as much impulse as that carried by the shock wave; the pressure profile of this wave at a point of interest in the field is called the first bubble pulse. Following rebound, the bubble expands again to a second maximum and then contracts until it collapses and rebounds again. This expansion–contraction–collapse–rebound sequence can repeat several times, each time with reduced amplitude.

Accompanying dilation is translation, in which the bubble migrates upward under the force of buoyancy. Eventually the bubble breaches, and the viewer is sometimes treated to a spectacular plume that shoots many meters skyward. In addition to dilation and translation, there is defomation. While the surface of the bubble remains essentially spherical when the bubble is large, it approaches a mushroom-cap profile as it collapses. In many cases, the deformation is so large that the bubble becomes a torus with a water jet shooting upward through the hole.

As just described, bubble motion well beneath the surface is characterized by three different time scales, with the times between dramatic events (shock-wave generation and bubble collapse/rebound) being one to two orders of magnitude greater than the duration of the events themselves. Also, there is a long list of possible energy repositories (kinetic and potential energy of the liquid, kinetic and potential energy of the bubble gas, gravitational potential energy due to buoyancy, potential energy due to surface tension) and loss mechanisms (acoustic radiation into the liquid, chemical reactions, molecular relaxation in the gas, heat and mass transfer at the bubble surface). As seen below, some of these are primary, some are secondary, and some are negligible. Finally, there are extensive experimental data available with which to compare theoretical predictions produced by various models. Much of these data are decades old, but are...
often the product of great skill and care. However, as discussed in Sec. VC, gaps remain.

**B. This study**

Shock-wave generation and bubble oscillation have often been treated separately. Here, we treat them comprehensively as two phases of a single phenomenon, with the first phase providing initial conditions to the second.

As discussed above, the shock-wave phase is short, typically less than a few milli-seconds. Extensive far-field pressure data have been obtained for the shock wave [Coles et al. (1946); Farley and Snay (1978); Price (1979)], but the authors know of no data that pertain to the inside or the surface of the highly compressed bubble. Recently, hydrocode simulations of detonation and initial bubble dynamics have been performed [Wardlaw and Mair (1998)]. Here, we employ a simple volume-acceleration model [Strasberg (1956); Frost and Harper (1975)] to determine shock-wave-phase bubble motion from a similitude relation for the far-field pressure profiles [Price (1979)].

Previous treatments of general bubble motion in the oscillation phase model the unbounded liquid as incompressible [Blake and Gibson (1987); Duncan and Zhang (1991a, b); Chahine and Kalumuck (1998)]. Here, we account for wave effects in the liquid by means of a first-order external doubly asymptotic approximation [Geers (1978); Geers and Zhang (1994a)]. The previous treatments of general bubble motion also ignore wave effects in the internal gas. We include such effects here through the use of a first-order internal doubly asymptotic approximation. Many equations of state have been developed for explosive gas products [Dobratz (1981)]. Two have been found suitable for the bubble oscillation phase [Jones and Miller (1948); Lee, Hornig, and Kurry (1968)], and have been consolidated into a single adiabatic equation of state for bubble oscillation.

The new bubble model is specialized to a spherical geometry for which only dilation and translation are considered. As discussed below, at least six equations exist for pure dilation of a spherical bubble in the oscillation phase [Lamb (1923); Herring (1941); Kirkwood and Bethe (1942); Keller and Kolodner (1956); Prosperetti and Lezzi (1986); Moss et al. (2000)]. The first treats the unbounded liquid as incompressible, while the next four introduce external wave effects through various means. The last addresses wave effects in both the external liquid and the internal gas. Extensions to modes beyond dilation have treated the liquid as incompressible [Penney and Price (1942); Taylor (1942); Ward (1943); Kolodner and Keller (1953); Hicks (1970)]. Here, we formulate equations for dilation plus translation that account for wave effects in both the liquid and the gas. The spherical-bubble equations for the new model are numerically integrated in time to produce dilation and translation histories. Agreement between these histories and previously obtained experimental data [Swift and Decius (1948); Snay (1962); Snay and Tipton (1963); Hicks (1970)] is substantially better than that produced by previous models.

**II. SHOCK-WAVE PHASE**

A dilational model of the bubble is appropriate for this brief, but important, phase. Extensive far-field pressure data have been obtained [Coles et al. (1946); Farley and Snay (1978); Price (1979)], and analytical models have been developed and validated against those data [Kirkwood and Bethe (1942); Cole (1948)]. The models are complicated and focus on far-field quantities; bubble-motion calculations are neither straightforward nor validated. Eulerian simulations have recently been performed [Wardlaw and Mair (1998)], but have not been validated by experimental data. To the authors’ knowledge, close-in experimental data are not available, apparently because of the inhospitable environment. We employ here a surprisingly useful volume-acceleration model to determine shock-wave-phase bubble motion from far-field pressure data.

**A. Similitude relations**

An accurate representation of far-field shock-wave pressure profiles is the similitude relation [Coles et al. (1946); Farley and Snay (1978); Price (1979)]

\[ P(R,t) = P_c [(a_c/R)^{1+A} f((a_c/R)^B v_c t/a_c)], \]

where \( R \) is the distance from the center of the explosive charge with radius \( a_c \), and \( P_c \), \( v_c \), \( A \), and \( B \) are constants associated with the charge material. Some recommended values for these constants appear in Table I. Two good choices for \( f(\tau) \) are

\[ f(\tau) = e^{-\tau}, \quad \tau \leq 1, \]

\[ f(\tau) = 0.8251 e^{-1.3387 \tau} + 0.1749 e^{-0.1805 \tau}, \quad \tau \leq 7. \]

The single-exponential fit does not extend to the start of the oscillation phase (Sec. III), so here we employ the double-exponential fit, which extends to the time when the pressure is down to about 5% of its peak.

A representative comparison of Eqs. (1) and (2) with a measured pressure profile is shown in Fig. 2 for the constants of Coles et al. (1946) in Table I. The fits within their respective ranges are good, with the exception of the small oscillation in the experimental profile. This oscillation is a mani-

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**Figure 1.** Typical bubble motion and free-field pressure history [Snay (1956)].
festation of wave effects in the bubble gas [Cole (1948); Wardlaw and Mair (1998)]. We constructed a high-order polynomial fit to the oscillation, but the effect on subsequent bubble motion was negligible. Hence, we regard Eq. (2b) as satisfactory for \( \tau \leq 7 \).

**B. Volume-acceleration model**

Now, we seek to use the far-field pressure profile given by Eq. (1) with \( f(\tau) \) given by Eq. (2b) to determine the motion of the bubble surface during the shock-wave phase. If the fields in the liquid were acoustic during this phase, we could use an equation proposed by Strasberg (1956), systematically formulated by Frost and Harper (1975), and discussed by Pierce (1989). This equation relates the far-field pressure profile to bubble motion as

\[
P(R,t) = \frac{\rho_l}{4\pi R} \bar{V}(t),
\]

where \( \rho_l \) is the mass density of the liquid and \( V \) is the bubble volume. Note that this profile expression does not contain the speed of sound in the liquid, \( c_l \), and is therefore independent of liquid compressibility. This is because the width of the wave in the liquid is substantially greater than the radius of the charge.

In our “hyperacoustic” situation, the bubble surface produces nonlinear wave propagation at speeds somewhat greater than \( c_l \). But, because Eq. (3) exhibits no dependence on \( c_l \), we might also expect negligible liquid-compressibility effects here. This is supported by the fact that \( A \) and \( B \) are both substantially less than 1, and that Eq. (1) describes acoustic wave propagation when \( A = B = 0 \). Hence, by comparison of Eqs. (1) and (3), we find that the appropriate extension of Eq. (3) to the hyperacoustic case is

\[
P(R,t) = \frac{\rho_l}{4\pi R} \left[ \frac{a_c}{R} \right]^A \bar{V}([a_c/R]^B t).
\]

Equating Eqs. (4) and (1) with \( f(\tau) \) given by Eq. (2b), we obtain for volume acceleration

\[
\bar{V}(t) = \frac{4\pi a_c}{\rho_l} P_c \left[ 0.8251 \exp(-1.338t/T_c) + 0.1749 \exp(-0.1805t/T_c) \right],
\]

where \( T_c = a_c/v_c \). Integration with \( \bar{V}(0) = 0 \) and further integration with \( V(0) = (4/3)\pi a_c^3 \) then yield

![FIG. 2. Fit of Eqs. (1) and (2) to a typical shock-wave pressure profile.](image-url)
conditions for subsequent bubble-response calculations during the oscillation phase and is given the symbol \( t_I \). Success of the partitioning concept depends, of course, on the insensitivity of subsequent response calculations to the choice of \( t_I \).

Figure 3 shows initial conditions computed from Eqs. (6) and (7) for \( t_I/T_c = 3, 4, 5, 6, \) and 7, and for three sets of charge constants [Coles et al. (1946); Farley and Snay (1978); Price (1979)], all pertaining to TNT. We see that \( a_I \) increases and \( a_\dot{I} \) decreases with increasing \( t_I \), as one would expect. It is shown below that these dependencies are such that, for a single set of charge constants, all pairs of initial-condition values for values of \( t_I \) between \( 3T_c \) and \( 7T_c \) produce essentially the same bubble response during the oscillation phase. This validates the partitioning concept.

### III. OSCILLATION PHASE: GENERAL MOTION

In view of the time scales mentioned in Sec. I A, we would expect to see incompressible flow of the unbounded liquid during the expansion–contraction segments of this phase and wave effects during the collapse–rebound segments. However, the earliest dilational bubble models as-
sumed only incompressible flow [Besant (1859); Lord Rayleigh (1917); Lamb (1923)], as did those for combined dilution and translation [Taylor (1942); Hicks (1970)], and those for general bubble motion [Penney and Price (1942); Ward (1943); Kolodner and Keller (1953)].

Even modern boundary-element models based on inviscid, irrotational flow of the liquid assume incompressible flow [Blake and Gibson (1987); Duncan and Zhang (1991a, b); Chahine and Kalumuck (1998)]. However, certain components of these models do not rely on that assumption. With \( \mathbf{R}_g(t) \) as the position vector for a gas particle at the bubble’s surface, \( \phi_g(\mathbf{R}_g) \) is the velocity potential at the particle’s location, \( \rho_g \) is the mass density of the internal gas, \( \rho_g \) is regarded as a known surface pressure, \( P_g \) is the uniform internal pressure defined by the equation of state, and \( d_{cg} \) is the depth of the bubble’s center of gravity. If \( \phi_g = 0 \), Eq. (9c) requires that \( p_g = P_g + \rho_g g(d - d_{cg}) \), which is the condition for static equilibrium.

The liquid and gas fields are linked at the bubble surface in two ways. First, compatibility of normal velocity requires that \( \mathbf{R}_g \cdot \mathbf{e}_n = \mathbf{R}_g \cdot \mathbf{e}_n \), i.e., \( \phi_g = \phi_g \), at every point on the surface. Second, pressure equilibrium requires that \( p_j = p_g - \sigma \kappa \) at every surface point, where \( \sigma \) is the surface tension and \( \kappa \) is the mean curvature. The effects of surface tension are negligible for underwater-explosion bubbles, so we take \( \sigma = 0 \).

We now formulate two bubble models. The first ignores wave phenomena by treating the external liquid as incompressible and the internal gas as having zero density. The second incorporates wave effects in both the liquid and the gas by means of first-order doubly asymptotic approximations.

### A. Waveless model

A three-dimensional boundary integral equation (BIE) that, for inviscid, irrotational, incompressible flow of the unbounded liquid, relates the velocity potential to its normal derivative over a smooth internal surface is [see, e.g., Banerjee and Butterfield (1981)]

\[
\phi_j = -H_j \phi_j,
\]

in which \( H_j = B_j^{-1} \Gamma_j \), where the geometry-dependent, boundary-integral operators \( B_j \) and \( \Gamma_j \) are defined by

\[
B_j \xi(\mathbf{R}_j) = \int_S [\mathbf{R}_j - \mathbf{R}_j]^{-1} \xi(\mathbf{R}_j') dS(\mathbf{R}_j'),
\]

\[
\Gamma_j \xi(\mathbf{R}_j) = \int_S [\mathbf{R}_j - \mathbf{R}_j]^{-3} (\mathbf{R}_j' - \mathbf{R}_j) \cdot \mathbf{n}_j(\mathbf{R}_j') \xi(\mathbf{R}_j') dS(\mathbf{R}_j').
\]

Here, \( \mathbf{n}_j \) is the unit normal to the surface \( S \), defined positive going into the liquid.

For \( \rho_g = 0 \), Eq. (9c) yields \( p_g = P_g \), so, for negligible surface tension, \( p_j = P_g \). Introducing Eq. (10) into Eq. (8b) and the result into Eq. (8a), and introducing \( p_j = P_g \) into Eq. (8c), we obtain the BIE-consolidated kinematic-compatibility and Bernoulli equations

\[
\mathbf{R}_g = -\mathbf{e}_n H_j \phi_j + \mathbf{e}_1 \phi_{g,1} + \mathbf{e}_2 \phi_{g,2},
\]

\[
\phi_j = \frac{1}{2} \mathbf{R}_g \cdot \mathbf{R}_g - \rho_j^{-1} \left[ P_g - (p_{atm} + \rho_g g) d \right].
\]
\( \mathbf{R}_t(t) = e, a_j \) and, from Sec. VA below, \( \phi_j(t) = -a_j \dot{a}_j \), where \( a_j = a_j(t) \) and \( \dot{a}_j = \dot{a}_j(t) \) are determined from Eqs. (7) and (6).

### B. DAA model

Now, Eq. (10) is quite accurate during the relatively slow expansion and contraction of the bubble. However, a boundary relation that is more applicable to bubble collapse and rebound is the plane-wave-radiation equation \( c_j \dot{\phi}_j = -\phi_{j,1} \), [Felippa (1980)], where \( \phi_{j,1} \) is the partial derivative of \( \phi_j \) with respect to time. The first-order differential equation that uniquely reduces to these two equations in their appropriate limits is the first-order external doubly asymptotic approximation (EDAA) \( \{ \text{Geers and Zhang (1994a)} \}\)

\[
\phi_j = -H_j \phi_{j,1} - c_j^{-1} \phi_{j,1,1}.
\]  

(13)

With \( \phi_{j,1,1} = \phi_j - \nabla \psi_j(\mathbf{R}_j) \cdot \nabla \psi_j(\mathbf{R}_j) = \phi_j - \mathbf{R}_j \cdot \mathbf{R}_j \), Eq. (13) becomes

\[
\phi_j = -H_j \phi_{j,1} - c_j^{-1} (\phi_j - \mathbf{R}_j \cdot \mathbf{R}_j).
\]  

(14)

This equation constitutes a simple means by which to include wave effects in the liquid during the bubble’s oscillation phase.

DAAs have been formulated in acoustics [Geers (1971, 1978); Nicolas-Vullierme (1991); Geers and Zhang (1994a, b); Geers and Toothaker (1997, 2000)]; electromagnetics [Geers and Zhang (1988)]; elastodynamics [Underwood and Geers (1981); Geers and Lewis (1997)]; and poroelastodynamics [Qi and Geers (1997)]. First-order DAAs were obtained by inspection [Geers (1971); Underwood and Geers (1981)]. Higher-order DAAs have been formulated by modal analysis [Geers (1978)] and by operator matching [Nicolas-Vullierme (1991); Geers and Zhang (1994a); Geers and Lewis (1997); Qi and Geers (1997); Geers and Toothaker (1997, 2000)]. Operator matching is akin to the method of matched asymptotic expansions [see, e.g., Van Dyke (1964)]. The formulation of a DAA for the present problem has been done by inspection.

Wave effects may be included in the gas model by means of the first-order internal doubly asymptotic approximation (IDAAs) \( \{ \text{Geers and Zhang (1994a)} \}\)

\[
\phi_j^e = H_{j} \phi_j^e + c_j^{-1} \phi_j^e_{,1},
\]  

(15)

in which \( H_{j} = B_{j}^{-1} \Gamma_j \), where \( B_j \) and \( \Gamma_j \) are given by Eqs. (11) with \( l = g \) and \( \mathbf{n}_j \) defined positive going into the gas. As discussed in Geers and Zhang (1994a), one may write \( \phi_j^e = \phi_j^e + \phi_j^e \), where \( \phi_j^e \), which is uniform over the bubble surface, pertains to dilational motion and \( \phi_j^e \), which has no uniform component, pertains to equivoluminal motion. It is shown by Geers and Zhang that \( H_{g} \phi_j^e = 0 \); hence, the second term in Eq. (15) is essential to admitting nonzero dilation.

With \( \psi_j^e = \psi_j - \nabla \psi_j(\mathbf{R}_g) \cdot \nabla \psi_j(\mathbf{R}_g) = \phi_j^e - \mathbf{R}_g \cdot \mathbf{R}_g \), Eq. (15) becomes

\[
\phi_j^e = H_{j} \phi_j^e + c_j^{-1} (\phi_j^e - \mathbf{R}_g \cdot \mathbf{R}_g).
\]  

(16)

To obtain the DAA-consolidated kinematic-compatibility equations, we eliminate \( \phi_j^e \) in Eq. (8b) by employing Eq. (14) and introduce the result into Eq. (8a), and then eliminate \( \phi_j^e \) in Eq. (9b) by employing Eq. (16) and introduce the result into Eq. (9a). This yields

\[
\dot{\mathbf{R}}_t = -e_n(H_j \phi_j + c_j^{-1} (\phi_j - \mathbf{R}_g \cdot \mathbf{R}_g)) + e_1 \phi_{j,1} + e_2 \phi_{j,2},
\]  

(17a)

\[
\dot{\mathbf{R}}_s = e_n(H_g \phi_g + c_g^{-1} (\phi_g - \mathbf{R}_g \cdot \mathbf{R}_g)) + e_1 \phi_{g,1} + e_2 \phi_{g,2}.
\]  

(17b)

We obtain the DAA-consolidated Bernoulli equations in four steps. In Eqs. (8c) and (9c), we eliminate \( \mathbf{V} \psi_j(\mathbf{R}_g) \) and \( \mathbf{V} \psi_j(\mathbf{R}_j) \) by means of Eqs. (8a) and (9a), respectively. Next, we solve the modified Eq. (8c) for \( p_t \) and the modified Eq. (9c) for \( p_g \), and equate the right sides of the resulting equations. Then, we enforce \( \phi_g^e = \phi_j^e \) by equating the right side of Eq. (16) to the right side of Eq. (14). Finally, we solve the resulting coupled algebraic equations for \( \phi_j \) and \( \phi_g \) to get

\[
\phi_j = (1 + \zeta)^{-1} \left[ \left( \frac{1}{2} + \frac{1}{2} \right) \frac{\rho_g}{\rho_t} \right] \mathbf{R}_g \cdot \mathbf{R}_g - \zeta c_j \left( H_j \phi_j \right)
\]

\[+ \left( 1 + \zeta \right) \rho_g \phi_g \]

\[\times \left[ \left( p_g^g - \left( p_{atm} + \rho_g d_{c_e} \right) \right) \left( \rho_t - \rho_g d_{c_e} \right) \right],
\]  

(18)

\[
\phi_g = (1 + \zeta)^{-1} \left[ \left( \frac{1}{2} + \frac{1}{2} \right) \frac{c_g}{c_t} \mathbf{R}_g \cdot \mathbf{R}_g + \left( \frac{\rho_g}{\rho_t} \frac{c_t}{c_g} \right) \mathbf{R}_g \cdot \mathbf{R}_g \right]
\]

\[\times \left[ \left( p_g^g - \left( p_{atm} + \rho_g d_{c_e} \right) \right) \left( \rho_t - \rho_g d_{c_e} \right) \right],
\]  

where \( \zeta = \rho_g c_t / \rho_t c_t \) is the specific-acoustic-impedance ratio. The right sides of Eqs. (18) are used for \( \phi_j \) and \( \phi_g \) on the right sides of Eq. (17). Equations (17) and (18) are new; they are amenable to numerical solution by boundary-element semidiscretization and explicit time integration, with iteration keyed to the dot products \( \mathbf{R}_j \cdot \mathbf{R}_j \) and \( \mathbf{R}_g \cdot \mathbf{R}_g \).

Regarding initial conditions at time \( t_i \), we again consider the bubble at \( t_i \) as spherical and centered at the origin. Then, the initial values are \( \mathbf{R}_g(t_i) = e, a_j \), and, from Sec. VB below, \( \phi_j(t_i) = \phi_{j,0}(t_i) + \phi_{j,1}(t_i) \cos \theta \) and \( \phi_g(t_i) = \phi_{g,0}(t_i) \cos \theta \), where \( \phi_{j,0}(t_i) \), \( \phi_{j,1} \) and \( \phi_{g,0}(t_i) \) are given by Eq. (72).

### C. Properties of the internal gas

There is a large body of work on the equation of state (EOS) for explosive gas products [Dobratz (1981)]. We excluded most EOS from consideration here, however, primarily because they do not consider explosives pertinent to underwater shock, and secondarily because we desired an adiabatic EOS for the gas. The retained EOS are due to Jones and Miller (1948), who produced adiabatic dense-gas EOS in tabular form, and to Lee, Hornig, and Kury (1968), who produced their dense-gas EOS by fitting an empirical relation to experimental data. The two EOS provide consistent models for the adiabatic expansion of TNT with mass densities in

\[J. \text{ Acoust. Soc. Am., Vol. 111, No. 4, April 2002}\]

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the range 1.00–1.63 g/cm³ and at pressures below 100 MPa. Hence, we shall use here the consolidated adiabatic equation of state

\[ P_g = K_c (V_c / V)^\gamma, \] (19)

where \( K_c \) is the adiabatic compression constant for the explosive material, \( V_c \) is the volume of the charge, \( V \) is the current volume of the bubble, and \( \gamma \) is the (constant) ratio of specific heats for the gas. Values of \( K_c \) and \( \gamma \) for TNT and Pentolite are given in Table II.

Wave effects in the gas constitute small perturbations (except in a small region surrounding the bubble center), so the material properties of the gas may be considered uniform throughout the volume of the bubble. Hence, conservation of mass yields for mass density

\[ \rho_g = \rho_c (V_c / V). \] (20)

The bulk modulus of the gas is defined by \( B_g = -\Delta P_g / (\Delta V / V) \). But, \( B_g = \rho_c c_g^2 \) and, from Eq. (19), \( \Delta P_g = -\gamma K_c (V_c / V)^\gamma (\Delta V / V) \). Hence,

\[ c_g = c_c (V_c / V)^{(1/2)(\gamma - 1)}, \] (21)

where \( c_c = \sqrt{\gamma K_c / \rho_c} \).

### IV. OSCILLATION PHASE: DILATION OF A SPHERICAL BUBBLE

For this simplest of cases, \( \mathbf{R}(t) = a(t) \mathbf{e}_z \), and \( \varphi(r,t) = \varphi(r,t) \) for both the liquid and the gas, where the origin of the coordinate system is at the center of the bubble, \( \mathbf{r}(t) \) is the position vector for a particle, and \( \varphi(r,t) \) is the velocity potential at the particle’s position. Hence, Eqs. (8a), (8b), (9a), and (9b) yield

\[ \dot{a} = \phi_t', \quad \ddot{a} = \phi_t'', \] (22)

and the averages of Eqs. (8c) and (9c) over the bubble surface give

\[ \dot{\phi}_t = \frac{1}{2} (\phi_t')^2 - \rho_t^{-1} (p_t - p_t'), \]
\[ \ddot{\phi}_t = \frac{1}{2} (\phi_t'')^2 - \rho_t^{-1} (p_g - P_g'), \] (23)

in which \( p_t = p_{\text{atm}} + \rho_g d_t \), where \( d_t \) is the fixed depth of the bubble’s center. Compatibility of normal velocity requires that \( \phi_t'' = \phi_t' \), which is also mandated by Eq. (22). Pressure equilibrium, for negligible surface tension, requires that \( p_t = p_g \).

### A. Waveless model

For dilation of a spherical surface with radius \( a \), the BIE, Eq. (10) becomes

\[ \phi_t' = -a^{-1} \phi_t. \] (24)

which is readily verified by noting that the pertinent solution to Laplace’s equation \( \varphi(r) = \varphi \cdot a / r \). Furthermore, Eq. (23b) with \( p_g = 0 \) yields \( p_g = P_g' \), so \( p_t = P_g \). Employing this and Eq. (22a) in Eq. (22a) and introducing Eq. (24) in Eq. (22a), we obtain

\[ \dot{a} = -a^{-1} \phi_t, \quad \ddot{a} = \frac{1}{2} a^{-2} - \rho_t^{-1} (p_g - p_t), \] (25)

where, from Eq. (19), \( P_g = K_c (a / L)^\gamma \). These equations correspond to Eq. (12). The initial conditions are \( a(t=0) = a_I \) and, from Eq. (25a), \( \phi_t(t=0) = -a_I \phi_t \).

We can eliminate \( \phi_t \) as a variable by multiplying Eq. (25a) through by \( a_I \), differentiating, and equating the resulting expression for \( \phi_t \) to the right side of Eq. (25b). This produces the following equation of motion (EOM) for the bubble radius:

\[ a \ddot{a} + \frac{3}{2} a^2 = \rho_t^{-1} (p_g - p_t). \] (26)

This is widely known as the Rayleigh–Plesset equation, but it was actually first derived by Lamb in 1923. We shall avoid misattribution and refer to it as the Lamb EOM. The initial conditions are simply \( a(t=0) = a_I \) and \( \dot{a}(t=0) = \dot{a}_I \).

### B. DAA model

With \( H \phi_t \), producing \( a^{-1} \phi_t \) for this case [see Eqs. (10) and (24)], and with \( H \phi_t = 0 \) for dilation, EDAA, Eq. (14), and IDAA, Eq. (16), become

\[ \phi_t' = -a^{-1} \phi_t - c_I^{-1} (\phi_t - a)^2, \quad \phi_t' = c_g^{-1} (\phi_g - a^2). \] (27)

The introduction of Eq. (22) then gives

\[ \dot{a} = -a^{-1} \phi_t - c_I^{-1} (\phi_t - a^2), \quad \ddot{a} = c_g^{-1} (\phi_g - a^2), \] (28)

which correspond to Eq. (17). Proceeding as described between Eqs. (17) and (18) with Eq. (8a)→Eq. (22a), Eq. (8c)→Eq. (23a), Eq. (9a)→Eq. (22b), Eq. (9c)→Eq. (23b), Eq. (14)→Eq. (27a), and Eq. (16)→Eq. (27b), we find

\[ \phi_t = (1 + \xi)^{-1} \left[ \frac{1}{2} + \frac{\rho_g}{p_t} \right]^2 - c_I \frac{\rho_t}{a} \phi_t - \rho_t^{-1} (p_g - p_t), \]
\[ \phi_g = (1 + \xi)^{-1} \left[ \frac{1}{2} + \frac{c_g}{c_I} \right]^2 - c_g \frac{\rho_t}{a} \phi_g + \rho_t^{-1} c_g (p_g - p_t), \] (29)

which correspond to Eq. (18). The right sides of Eq. (29) are used for \( \phi_t \) and \( \phi_g \) on the right sides of Eq. (28). Actually, there is no need to consider \( \phi_g \), because Eqs. (28a) and (29a) are sufficient to calculate \( a(t) \).

Three initial values are required by Eqs. (28a) and (29a). The first two are again \( a(t=0) = a_I \) and \( \dot{a}(t=0) = \dot{a}_I \). To calculate \( \phi_t(t=0) \), we solve Eq. (28a) for \( \phi_t \) and equate the right side of the result to the right side of Eq. (29a); or, we solve Eq. (28b) for \( \phi_g \) and equate the right side of that result to the right side of Eq. (29b). Either procedure yields

### TABLE II. Values of \( K_c \) and \( \gamma \) appearing in the consolidated adiabatic EOS.

<table>
<thead>
<tr>
<th>Material</th>
<th>Source</th>
<th>( K_c ) [MPa]</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TNT (1.00 g/cc)</td>
<td>Jones and Miller</td>
<td>556.2</td>
<td>1.20</td>
</tr>
<tr>
<td>TNT (1.50 g/cc)</td>
<td>Jones and Miller</td>
<td>839.6</td>
<td>1.27</td>
</tr>
<tr>
<td>TNT (1.63 g/cc)</td>
<td>Lee et al. (1973)</td>
<td>1045</td>
<td>1.30</td>
</tr>
<tr>
<td>Pentolite (1.70 g/cc)</td>
<td>Lee and Walton (1975)</td>
<td>1033</td>
<td>1.35</td>
</tr>
</tbody>
</table>
\[ \phi_l = -a\dot{a} \left[ 1 + \dot{\xi} - \frac{1}{2} (1 - \rho_s / \rho_t) \frac{\dot{a}}{c_l} \right] + \frac{a}{\rho_t c_l} (P_g - p_t), \]  

which we evaluate at \( t = t_f \). If interested, we may calculate \( \phi_g \) from Eq. (29b), with \( \phi_g(t_f) \) specified as any constant, zero being the most convenient.

Here too, we can eliminate \( \phi_t \) as an unknown. We introduce Eq. (30) into Eq. (29a) to get

\[ \phi_l = \frac{1}{2} (1 + \rho_s / \rho_t) \dot{a}^2 + \frac{\dot{\xi}}{c_s} \dot{a} - \rho_l^{-1} (P_g - p_t). \]  

Then, we differentiate Eq. (30) and equate the right side of the result to the right side of Eq. (31). This yields the DAA EOM

\[ a\dot{a} \left[ 1 + \dot{\xi} - \frac{1}{2} (1 - \rho_s / \rho_t) \frac{\dot{a}}{c_l} \right] + \frac{3}{2} \dot{a}^2 \left[ 1 + \frac{\dot{\xi}}{3} \frac{\dot{a}}{a} \right] + \frac{1}{3} \left( \frac{\rho_s}{\rho_t} \right) \left[ 1 + \frac{\dot{\xi}}{c_s} \frac{\dot{a}}{c_l} \right] + \frac{\rho_s}{\rho_t} \frac{\dot{a}}{c_l} \left( \rho_s \frac{\dot{a}}{c_l} \right) a \dot{a} + \dot{\xi} \left( c_s + \frac{\dot{a}}{c_l} \right) \dot{a} = \rho_l^{-1} (P_g - p_t) \left[ 1 + \frac{\dot{a}}{c_l} \right] + \frac{a}{c_l} P_g, \]  

with initial conditions \( a(t_f) = a_l \) and \( \dot{a}(t_f) = \dot{a}_l \). This new equation of motion accounts for wave effects in both the liquid and the gas, and reduces to Eq. (26) for \( c_l \rightarrow \infty, \rho_t \rightarrow 0 \).

Equation (32) provides greater insight if it is “normalized” to Eq. (26) with, from Eqs. (19)–(21)

\[ P_g = -3 \rho_g c_s^2 \frac{\dot{a}}{a}, \quad \rho_g = -3 \rho_s \frac{\dot{a}}{a}, \quad \dot{a} = -\frac{1}{2} (\gamma + 1) \frac{\dot{\xi}}{\dot{a}}. \]  

Introducing these into Eq. (32) and neglecting terms in \((\dot{a}/c_l)^2\), we obtain

\[ a\dot{a} \left[ 1 + \dot{\xi} - \frac{2}{3} \frac{\rho_s}{\rho_t} \frac{\dot{a}}{c_l} \right] + \frac{3}{2} \dot{a}^2 \left[ 1 - \frac{1}{2} (\gamma + 1) \frac{\dot{\xi}}{\dot{a}} \right] + \frac{1}{3} \left( \frac{\rho_s}{\rho_t} \right) \left[ 1 + \frac{\dot{\xi}}{c_s} \frac{\dot{a}}{c_l} \right] + \frac{\rho_s}{\rho_t} \frac{\dot{a}}{c_l} \left( \rho_s \frac{\dot{a}}{c_l} \right) a \dot{a} + \dot{\xi} \left( c_s + \frac{\dot{a}}{c_l} \right) \dot{a} = \rho_l^{-1} (P_g - p_t). \]  

Comparing this with Eq. (26), we see that wave effects introduce modest response dependence into the coefficients of \( a\dot{a} \) and \( \dot{a}^2 \). However, their principal impact is to introduce two damping terms, with waves in the gas and liquid producing the coefficients \( \dot{\xi} c_l = (\rho_s / \rho_t) c_s \) and \( 3\dot{\xi} c_s = 3\rho_s c_s^2 / \rho_t c_l \), respectively. It is seen below that these \( a \)-dependent coefficients are comparable in size. Finally, it is interesting to note that, from Eq. (33c), the term \( \dot{\xi} a\dot{a} \) in Eq. (32) produces an \( \dot{a}^2 \) term in Eq. (34), and, from Eq. (33a), the term \( \rho_l^{-1} (a/c_l) P_g \) in Eq. (32) produces the term \( 3 \dot{\xi} c_s \dot{a} \) plus an \( \dot{a}^2 \) term in Eq. (34).

### C. Mixed and previous models

There are two models that lie between the waveless and DAA models. The first accounts for wave effects in the liquid but not in the gas; the pertinent EOM is Eq. (32) with \( \rho_g = \rho_s = \dot{\xi} = \dot{x} = 0 \). The second accounts for wave effects in the gas but not in the liquid; the pertinent EOM is Eq. (32) with \( \dot{\xi} = 0, c_s \rightarrow \infty \), and \( \dot{\xi} c_l = (\rho_s / \rho_t) c_s \).

Table III shows some earlier dilational EOM that account for wave effects in the liquid but not in the gas. We see that the EOM for our first mixed model is identical to that of Keller and Kolodner (1956). In fact, if we divide each of the equations in Table III through by the coefficient of \( a\dot{a} \), we obtain the generic EOM

\[ a\dot{a} + \frac{\dot{a}^2}{3} \left[ 1 + \frac{2}{3} \frac{\dot{a}}{c_l} + O_2 \right] = \rho_l^{-1} \left( P_g - p_t \right) \left[ 1 + \frac{2}{3} \frac{\dot{a}}{c_l} + O_2 \right] + \frac{a}{c_l} P_g \left( 1 + O_1 \right), \]  

where \( O_1 \) and \( O_2 \) denote terms in first and second powers, respectively, of the dilational Mach number \( a / c_l \). In the following, we refer to the EOM for the first mixed model as the K&K EOM and refer to the EOM for the second mixed model as the NPC (not previously considered) EOM.

It is informative to observe the effect of taking \( \phi_t, \) \( \phi_l \in \text{EDAA}_1 \), i.e., neglecting \( \dot{a}^2 \) in Eq. (28a). The resulting EOM is the K&K EOM with the signs of the Mach-number terms on the left side reversed. Although this EOM performs satisfactorily in response computations, it is not consistent with the generic EOM, Eq. (35).

In 2000, Moss, Levatin, and Szeri (MLS) published a model for internal wave effects in a spherical bubble that
calls for merely replacing \( P_g \) with \( P_g + (a/3c_g)\dot{P}_g \). With \( V/JV = (a/\dot{a})^3 \) in Eqs. (19)–(21), it is easily found that
\[
(a/3c_g)\dot{P}_g = -\rho_{cg} \dot{a}.
\]
Hence, the MLS modification corresponds to neglecting the convection terms in both Bernoulli’s equation, Eq. (23b), and IDAA\(_1\), Eq. (28b), for the gas, which yields \( P_g = \rho_{cg} \dot{a} \). Introduction of the MLS replacement into the K&K EOM yields
\[
a\ddot{a} = \left( 1 + \frac{\dot{a}}{c_l} \right)^\gamma \left[ 1 - (\gamma + \frac{1}{\gamma}) \xi - \frac{1}{3} \left( \frac{\dot{a}}{c_l} \right) \right] \frac{\dot{a}}{c_l}^{\gamma - 1} + \frac{1}{c_l} \frac{\dot{a}}{c_l} \frac{\dot{P}_g}{\rho_{cg}},
\]
(36)
to which Eq. (32) reduces if \( (\rho_g/\rho_l)\dot{a}/c_l \ll 1 \) and \( \rho_g/3\rho_l \ll 1 \). If one accepts these restrictions in order to simplify Eq. (32), one might also take \( \xi = 0 \) in Eq. (36), which yields a very simple equation. However, we prefer to stay with Eq. (32), which imposes no restrictions beyond those underlying the systematic formulation.

**D. Computational versus experimental results for dilation**

In this section we show nondimensional bubble-radius histories computed with the shock-wave ICs of Sec. II C, various equations of motion, and the equation of state given by Eq. (19). Pressure is normalized to \( P_l \), and length and time are normalized to
\[
\ddot{a} = a_\ell(K_c/p_l)^{1/3}, \quad \tau = \sqrt{\tau_0/p_l}.
\]
(37)
Based on \( P_g = K_c(a_/\dot{a})^{3\gamma} \), \( \ddot{a} \) is the adiabatic equilibrium radius, i.e., the static radius at which \( P_g = p_l \). With this parametrization, the nondimensional Lamb EOM [see Eq. (26)]
is
\[
\ddot{a} + \frac{1}{\ddot{a}} = a^{3\gamma - 1},
\]
where \( \ddot{a} = a/\dot{a} \) and a prime denotes differentiation with respect to \( t = t/t_0 \).

In 1948, Swift and Decius reported results from an extensive array of underwater explosion tests, among which were nine shots, each for a nominally 300 g charge of TNT with mass density 1.50 g/cm\(^3\) detonated at a depth of 92 m. This depth was sufficiently large that each bubble remained essentially spherical and translated less than \( 1/10 \) of its depth over three bubble periods. We shall use this ensemble of nearly identical shots as our benchmark test.

Shown in Fig. 4 are bubble-radius histories produced for the benchmark test by the Lamb and DAA EOM for \( t_1/T_c = 7 \), the 1.50 g/cm\(^3\) TNT constants from Table II, and the shock-wave-phase charge constants of Price (1979); Farley and Snay (1978); and Coles et al. (1946). Also shown as horizontal (vertical) lines are the maximum radii (times of minimum radii) measured by Swift and Decius. First, we note that the Lamb EOM erroneously produces no response decay and the DAA EOM correctly produces substantial response decay. Second, we observe that the DAA EOM predicts within 10\% the first radial maximum and first minimum-radius time for the shock-wave-phase charge constants of Price and of Coles et al., but not for the constants of Farley and Snay.

We investigated the charge-constant discrepancies by examining post-shock-wave energy associated with DAA-EOM calculations at time \( t_1 \) for the five choices \( t_1/T_c = 3, 4, 5, 6, \) and 7. As mentioned above, the flow at these times is nearly lossless; hence, we can calculate the total energy at the beginning of the oscillation phase from Cole (1948)
\[
E_1 = 2 \pi \rho_{sg} a_1^2 \ddot{a}_1^2 + \frac{1}{2} \pi \rho_{sg} a_1^2 + \frac{1}{2} \pi (\gamma - 1)^{-1} K_c(a_/a_1)^{3\gamma} a_1^2.
\]
(39)
The first term on the right is the kinetic energy of incompressible liquid, the second is potential energy due to buoyancy, and the third is potential energy in the bubble gas. Table IV shows statistical data from the five calculations for

**TABLE IV. Post-shock-wave energy from three sets of TNT charge constants.**

<table>
<thead>
<tr>
<th>Data set</th>
<th>Average (KJ)</th>
<th>Std. deviation (%) of average</th>
<th>Percent of original</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coles et al.</td>
<td>604</td>
<td>1.86%</td>
<td>42.2%</td>
</tr>
<tr>
<td>Farley and Snay</td>
<td>550</td>
<td>3.10%</td>
<td>38.4%</td>
</tr>
<tr>
<td>Price (1979)</td>
<td>661</td>
<td>1.25%</td>
<td>46.2%</td>
</tr>
</tbody>
</table>
each set of shock-wave-phase charge constants: $P_c$, $v_c$, $A$, and $B$. The small standard deviations demonstrate that the choice of $t_I$ over the range considered is unimportant. For a detonation energy of 4770 J/g for TNT of density 1.5 g/cm$^3$ [Cook (1968)], the three charge-constant sets yield the entries in the last column of the table. Only Price’s set yields a value for the percent of original energy left for the oscillation phase that lies within the range of previous estimates, viz., 45%–50% [Taylor (1942), Arons and Yennie (1948)], although the set of Coles et al. comes close. We use Price’s shock-wave-phase charge constants in all subsequent calculations.

Figure 5 shows computed bubble-radius histories for the benchmark test produced by the DAA, K&K, and NPC EOM for $t_I/T_c$ = 3, 5, and 7. Also shown are the measured maximum radii and minimum-radius times from Fig. 4. The closeness of each trio of curves for a specified EOM again demonstrates the insensitivity of bubble response to $t_I$ values within the range $3T_c \leq t_I \leq 7T_c$. We see in the figure that all the histories exhibit values for first bubble maximum and first minimum-radius time that agree satisfactorily with the experimental data. However, the K&K and NPC histories do not exhibit satisfactory values for the second bubble maximum, in contrast to the DAA histories. The DAA EOM also outperforms the K&K and NPC EOM in predicting the third bubble maximum and minimum-radius time, but the DAA histories do not produce particularly accurate values. The NPC and K&K histories show energy loss in the gas as comparable to energy loss in the liquid, as mentioned in the discussion following Eq. (34). We also computed response histories with Eq. (36); they are barely distinguishable from the DAA histories. Finally, the nearly coincident K & K histories in Fig. 5 resemble closely a corresponding history presented in Fig. 5 of Chisum and Shin (1997), which was generated with a hydrocode.

It is informative to examine the ratios $\rho_g/\rho_l$, $c_g/c_l$, $\zeta = \rho_g c_g^2/\rho_l c_l^2$, $\bar{a} c_g$, and $\bar{a} c_l$ over the three bubble periods of Fig. 5. These ratios are shown in Fig. 6 as produced by the DAA EOM for $t_I/T_c$ = 7. We see that all five ratios remain substantially smaller than unity. That $(\bar{a}/c_g)^2 \ll 1$ and $(\bar{a}/c_l)^2 \ll 1$ is reassuring, because the DAA EOM is accurate only when both of these limitations hold, just as the EOM...
listed in Table III are accurate only for \((a/c_g)^2 \leq 1\) [Prosperetti and Lezzi (1986)]. Figure 6 shows that the largest value of \((a/c_g)^2\) occurs at \(t = t_f\); that value is 0.11. The same holds true for \(t_f = 5T_c\), but the largest value is 0.19. Hence, the best available selection for \(t_f\) is \(7T_c\), which is the value for which the DAA EOM performs best in Fig. 5.

E. Energy-loss mechanisms

Figure 5 implies that wave effects in the external liquid and the internal gas are the primary energy-loss mechanisms involved. It’s obvious that the wave energy in the liquid is lost by acoustic radiation to the far field. But, the wave energy in the gas is internal, and can be lost only by transmission into the liquid for subsequent radiation or by conversion into another form. We consider the transmission mechanism first.

Figure 7 shows total energy as a function of time calculated from Eq. (39) with \(a_I\) replaced by \(a(t)\). For the Lamb EOM, Eq. (26), \(E(t)\) is constant because there is no energy-loss mechanism. There are two such mechanisms, however, for the DAA EOM, Eq. (32), so \(E(t)\) decreases with time. As one would expect, however, the decrease is not steady, but consists of abrupt drops at times when the radial velocity is high. At each of these times, two spherical acoustic pulse waves are generated, one propagating outward and one propagating inward. Roughly speaking, the inward-propagating pulse has the pressure profile of a sine wave with a duration of one period \(T_p\), and the outward-propagating pulse has the negative of that profile. Let us examine the propagation of the internal pulse wave generated by the bubble’s first collapse/rebound. Because \(\tilde{a}\) is always substantially smaller than \(c_g\) (Fig. 6), we neglect Doppler effects.

The internal pulse wave propagates with velocity \(c_g\), which, from Eq. (21), varies in accordance with time-dependent bubble volume. The pulse travels inward until it reaches the bubble center and then returns to the bubble surface at time \(t_{m1} + t_1\), where \(t_{m1}\) is the time of first collapse/rebound and \(t_1\) is determined from

\[
\int_{t_{m1}}^{t_{m1} + t_1} c_g(t) dt = a(t_{m1}) + a(t_{m1} + t_1).
\]

Upon reaching the bubble surface, the pulse undergoes reflection and transmission, with energy fraction \(f_1\) being reflected back into the bubble gas. The reflected pulse travels inward with variable velocity \(c_g\), returns when it reaches the bubble center, and reaches the bubble surface at the time \(t_{m1} + t_1 + t_2\), where \(t_2\) is determined from

\[
\int_{t_{m1} + t_1}^{t_{m1} + t_1 + t_2} c_g(t) dt = a(t_{m1} + t_1) + a(t_{m1} + t_1 + t_2).
\]

At this time, a second reflected pulse with energy fraction \(f_2\) travels inward, and the events repeat. Let us follow the process through time \(t_f\) determined by

\[
t_{m1} + \sum_{j=1}^{J} t_j < t_M2 \quad \text{and} \quad t_{m1} + \sum_{j=1}^{J+1} t_j > t_M2,
\]

where \(t_{M2}\) is the time of second bubble maximum. The fraction of first-collapse/rebound wave energy remaining at the time of second bubble maximum is then

\[
F_j = \prod_{j=1}^{J} f_j.
\]

If the characteristic wavelength, \(c_g T_p\), were much smaller than \(\tilde{a}\), where \(c_g\) is a characteristic value of \(c_g\) during \(t_{m1} \leq t \leq t_{M2}\), then transmission and reflection at the bubble surface would be nearly planar. This would produce an energy fraction reflected back into the bubble at time \(t_f\) of approximately [Kinsler et al. (1982)]

\[
f_j = \left[\frac{1 - \zeta(t_{m1} + t_j)}{1 + \zeta(t_{m1} + t_j)}\right]^2.
\]

However, the DAA-EOM calculations for the benchmark test produced a characteristic wavelength greater than \(\tilde{a}\), in which case the reflected energy fractions produced by Eq. (44) constitute lower bounds. Equation (42) for the benchmark test gives \(J = 8\), and Eq. (43) with Eq. (44) yields \(F_j = 0.91\). Hence, the transmission of internal wave energy out.
to the liquid is an insignificant loss mechanism. This is because, as seen in Fig. 6, the specific-acoustic-impedance ratio $\zeta$ is very small at all times.

Regarding internal wave-energy loss by conversion to another form, we recall the convergence of an internal spherical pulse wave at the bubble center. Theoretically, the energy density in the pulse wave becomes unbounded there; actually, the transient pressures and temperatures approach levels far in excess of those characterizing the waveless gas model. These levels admit energy-loss mechanisms other than acoustic, such as gas dissociation and chemical reaction [Suslick (1990); Colussi et al. (1998)]. Although a study of such mechanisms is beyond the scope of this paper, it is clear that wave convergence offers ample opportunity for the loss of internal wave energy.

Even though the DAA radial-displacement histories in Fig. 5 indicate that wave effects in the external liquid and the internal gas are the primary energy-loss mechanisms, the histories do display a rate of decay that is somewhat slower than that seen in the tests, which implies that there is at least one other secondary mechanism at work. One secondary mechanism, heat and/or mass transfer at the bubble surface, would invalidate the adiabatic equation of state, Eq. (19) [see, e.g., Hentschel and Lauterborn (1982)]. A symptom of this type of energy loss is a bubble-radius history that settles down to a late-time asymptotic radius $a_\infty$ smaller than the equilibrium radius $\bar{a}$. However, this symptom is not observed in Fig. 8, which shows measured bubble-radius histories for TNT and Tetryl charges [Swift and Decius (1948)] that are settling down nicely to their respective equilibrium radii. A contrary conclusion would be drawn from Fig. 8.1 in Cole (1948), which shows a measured bubble-radius history approaching a late-time asymptote of about 8 in., one-third smaller than the equilibrium-radius value drawn on the figure. However, we calculated for this experiment a value for equilibrium radius of 8 in., not 12 in.

Another possible energy-loss mechanism is the transfer of dilational energy to deformational modes of response. This has been considered by Penney and Price (1942); Ward (1943); Kolodner and Keller (1953); and recently by Longuet-Higgins (1992), Feng and Leal (1993), Mao et al. (1995), and McDonald and Holland (2000). These researchers have shown that deformation can grow rapidly during bubble contraction, with computations typically failing at collapse. On the other hand, experiments have shown the bubble regaining its sphericity during subsequent expansion. This complex behavior is presently a topic of active research.
V. OSCILLATION PHASE: DILATION PLUS TRANSLATION OF A SPHERICAL BUBBLE

From Fig. 9, the position of an azimuthal ring of liquid particles on the spherical bubble surface may be described as

$$\mathbf{R}(t) = [a(t) + u(t) \cos \vartheta(t)] \mathbf{e}_r [\vartheta(t)] - u(t) \sin \vartheta(t) \mathbf{e}_\vartheta [\vartheta(t)].$$  \hspace{1cm} (45)

Hence,

$$\dot{\mathbf{R}}(t) = (\dot{a} + \dot{u} \cos \vartheta(t)) \mathbf{e}_r (\vartheta(t)) + [a \dot{\vartheta}(t) - \dot{u} \sin \vartheta(t)] \mathbf{e}_\vartheta (\vartheta(t)).$$  \hspace{1cm} (46)

where the arguments of the time-dependent functions on the right have been dropped. From symmetry, we write $a \dot{\vartheta}(t) = \dot{v}_r \sin \vartheta(t)$, so that Eq. (46) becomes

$$\dot{\mathbf{R}}(t) = (\dot{a} + \dot{u} \cos \vartheta(t)) \mathbf{e}_r (\vartheta(t)) + (\dot{v}_r - \dot{u}) \sin \vartheta(t) \mathbf{e}_\vartheta (\vartheta(t)).$$ \hspace{1cm} (47)

Now at any instant, the velocity potential in the liquid, and its gradient, may be expressed in axisymmetric spherical coordinates for $r = a(t)$ as (Fig. 9)

$$\varphi_r(r, \theta, t) = \varphi_{0r}(r, t) + \varphi_1(r, t) \cos \theta,$$ \hspace{1cm} (48a)

$$\nabla \varphi_r(r, \theta, t) = [\varphi_{0r}(r, t) + \varphi_1(r, t) \cos \theta] \mathbf{e}_r (\theta) - r^{-1} \varphi_1(r, t) \sin \theta \mathbf{e}_\theta (\theta).$$ \hspace{1cm} (48b)

Therefore, we can enforce Eq. (8a) by equating the right side of Eq. (47) to the right side of Eq. (48b) expressed at $r = a(t)$, $\theta = \vartheta(t)$, which yields

$$\dot{a} + \dot{u} \cos \vartheta = \varphi_0 \dot{a} + \varphi_1 \dot{u},$$ \hspace{1cm} (49a)

$$(\dot{v}_r - \dot{u}) \sin \vartheta = -a^{-1} \varphi_1 \dot{u},$$ \hspace{1cm} (49b)

where $\varphi_0(a(t)) = \varphi_{0r}(a(t))$ and $\varphi_1(a(t)) = [\varphi_{0r}(a(t)) \varphi_1(a(t))]_{r=\text{at}}$. Now, the functions 1 and $\varphi_1$ are, with weight-function $\sin \vartheta$, for the spherical geometry, mutually orthogonal over the domain $0 \leq \vartheta \leq \pi$. Hence, we project Eq. (49a) onto each of those functions by multiplying both sides, first by $\sin \vartheta$ and then by $\sin \vartheta \cos \vartheta$, and integrating the two resulting equations over $0 \leq \vartheta \leq \pi$. Then, we project Eq. (49b) onto the function $\sin \vartheta$ by multiplying both sides by $\sin^2 \vartheta \sin \vartheta$ and integrating the resulting equation over $0 \leq \vartheta \leq \pi$. This yields

$$\dot{a} = \varphi_0^{(a)}, \quad \dot{u} = \varphi_1^{(a)}, \quad \dot{v}_r = \varphi_1^{(a)} - a^{-1} \varphi_1(\vartheta).$$ \hspace{1cm} (50)

Next, using Eq. (48) expressed at $r = a(t)$, $\theta = \vartheta(t)$, we write Bernoulli’s equation at the bubble surface, Eq. (8c), as

$$\phi_0 + \varphi_1 \cos \vartheta - \varphi_1 \dot{\vartheta}(\vartheta(t)) \sin \vartheta = \frac{1}{2} [(\phi_0 + \varphi_1 \cos \vartheta)^2 + (a^{-1} \varphi_1 \vartheta)^2 \sin^2 \vartheta] - \rho_1^{-1} [p_{0r} + p_{1r} \cos \vartheta] - [P_{\text{am}} + \rho_\text{g}(d_1 - u - a \cos \vartheta)].$$ \hspace{1cm} (51)

From $a \dot{\vartheta}(\vartheta(t)) = \dot{v}_r \sin \vartheta(t)$ and Eq. (50c), we find $\dot{\vartheta}(\vartheta(t)) = a^{-1} (\varphi_1^{(a)} - a^{-1} \varphi_1^{(a)}) \sin \vartheta(t)$, which we introduce into Eq. (51). Then, we project the resulting expression, first onto the function 1 (dilational projection) and then onto the function $\cos \vartheta$ (translational projection), to obtain

$$\phi_1(\vartheta(t)) = \frac{1}{2} (\phi_0^{(a)})^2 + \frac{1}{2} (\varphi_1^{(a)})^2 + \frac{1}{2} a^{-1} \varphi_1(2 \phi_1 - a^{-1} \varphi_1) - \rho_1^{-1} [p_{0r} + p_{1r} \varphi_1(\vartheta)] s \varphi_1 \vartheta(t) (\vartheta(t)) - \rho_1^{-1} [p_{0r} + p_{1r} \varphi_1(\vartheta)].$$ \hspace{1cm} (52)

Proceeding similarly with $r \leq a(t)$ for the internal gas, we obtain

$$\dot{a} = \varphi_0^{(g)}, \quad \dot{u} = \varphi_1^{(g)}, \quad \dot{v}_r = \varphi_1^{(g)} - a^{-1} \varphi_1^{(g)},$$ \hspace{1cm} (53)

and

$$\phi_0^{(g)} = \frac{1}{2} (\phi'_0)^2 + \frac{1}{2} (\phi'_1)^2 + \frac{1}{2} a^{-1} \phi_1(2 \phi'_1 - a^{-1} \phi_1) - \rho_0^{-1} (p_{0r} - P_{g}),$$ \hspace{1cm} (54)

The velocity-compatibility and pressure-equilibrium conditions are $[\varphi_0^{(g)} + \varphi_1^{(g)} \cos \vartheta(\vartheta(t))]_{\vartheta(\vartheta(t)) = \vartheta(\vartheta(t))} = \varphi_0^{(g)} + \varphi_1^{(g)} \cos \vartheta(\vartheta(t))$ and $[p_{0r} + p_{1r} \cos \vartheta(\vartheta(t))]_{\vartheta(\vartheta(t)) = \vartheta(\vartheta(t))} = p_{0r} + p_{1r} \cos \vartheta(\vartheta(t))$, respectively. Projection then yields $\phi_0^{(g)} = \phi_0^{(g)}, \phi_1^{(g)} = \phi_1^{(g)}, p_{0r} = p_{0r},$ and $p_{1r} = p_{1r}$. 

A. Waveless model

For dilution plus translation of a spherical surface with radius $a$, the BIE, Eq. (10), becomes

$$\phi_0^{(a)} + \phi_1^{(a)} \cos \vartheta = -a^{-1} (\phi_0^{(a)} + 2 \phi_1^{(a)} \cos \vartheta),$$ \hspace{1cm} (55)

which is readily verified by noting that the pertinent solution to Laplace’s equation is $\varphi_r(r, \theta, t) = \varphi_0^{(a)}(t) \frac{a}{r} \varphi_1^{(a)}(t) \cdot (a/r)^2 \cos \theta$. Dilational and translational projection of Eq. (55) gives

$$\phi_0^{(a)} = -a^{-1} \phi_0^{(a)}, \quad \phi_1^{(a)} = -2 a^{-1} \phi_1^{(a)}.$$ \hspace{1cm} (56)

Using Eqs. (50a) and (50b) to eliminate $\phi_0^{(a)}$ and $\phi_1^{(a)}$ from these relations, we find

$$\dot{a} = -a^{-1} \phi_0^{(a)}, \quad \dot{u} = -2 a^{-1} \phi_1^{(a)},$$ \hspace{1cm} (57)

which correspond to Eq. (12a).

With $p_{0r} = P_r$ and $p_{1r} = 0$, pressure equilibrium yields $p_{0r} = P_r$ and $p_{1r} = 0$. Introducing these and Eqs. (56) into Eqs. (52), we obtain

$$\phi_0^{(a)} = \frac{1}{2} (a^{-1} \phi_0^{(a)})^2 - (a^{-1} \phi_1^{(a)})^2 - \rho_1^{-1} (P_r - p_{0r} + \rho_\text{g} u),$$ \hspace{1cm} (58a)

$$\phi_1^{(a)} = 2 (a^{-1} \phi_0^{(a)})^2 - (a^{-1} \phi_1^{(a)})^2 - \rho_1^{-1} (P_r - p_{0r} + \rho_\text{g} u),$$ \hspace{1cm} (58b)

which correspond to Eq. (12b). The initial conditions for Eqs. (57) and (58) are $a(t_i) = a_i, \quad u(t_i) = 0, \quad \phi_0(t_i) = -a_i \dot{a}_i$, and $\phi_1(t_i) = 0$, the latter pair coming from Eq. (57) and $\dot{u}(t_i) = 0$.

We can eliminate $\phi_0^{(a)}$ and $\phi_1^{(a)}$ as variables in three steps. First, employ Eq. (57) to eliminate them on the right sides of Eq. (58). Second, multiply Eq. (57a) by $a$, differentiate to get an expression for $\phi_0^{(a)}$, and equate that expression to the right side of the modified Eq. (58a). Third, multiply Eq. (57b) by $a$, differentiate to get an expression for $\phi_1^{(a)}$, and equate that expression to the right side of the modified Eq. (58b). This yields the following waveless EOM for dilution plus translation:
These agree closely with previous equations obtained by different means. Penney and Price (1942) and Ward (1943) obtained equations that differ from Eq. (59) only in the absence of the term $\frac{1}{2}a^2$. Hicks (1970) used Lagrange’s equation to obtain Eq. (59) in its entirety. Hence, we shall refer to them as the Hicks EOM. The initial conditions for Eq. (59) are $a(t_I) = a_I, \dot{a}(t_I) = \dot{a_I}$, $u(t_I) = 0$, and $\ddot{u}(t_I) = 0$. The last of these, when introduced into Eq. (59b), leads to $u(t) = g(t - t_I)^2$ at very early time.

It is informative to consider Eq. (59) from the standpoint of generalized momenta and forces. The former are defined as $P_a = \dot{a}T/\dot{a}$, where $T$ is system kinetic energy and $q$ is a generalized displacement [Meirovitch (1997)]. Here, the momenta are given by

$$P_a = 4\pi \rho_a a^3 \dot{a}, \quad \dot{P}_a = \frac{3}{2} \pi \rho_a a^3 \dot{u},$$

i.e., each is the product of an “added mass” of liquid [Milne-Thomson (1960)] and its associated velocity. We may write Eq. (59) in terms of these quantities as

$$\begin{align*}
\ddot{P}_a &= F_{gas} - F_{amb} + F_{exp}, \\
\dot{P}_a &= F_{buoy},
\end{align*}$$

where the (always positive) gas, ambient, expansion, and buoyancy forces are given by

$$\begin{align*}
F_{gas} &= 4\pi a^2 K_e(a/c)^3 \gamma - 2, \\
F_{amb} &= 4\pi a^2 [p_{amb} + \rho g (d_l - u)], \\
F_{exp} &= \pi a^2 (6a^2 + u^2), \\
F_{buoy} &= \frac{1}{2} \pi \rho_g a^3.
\end{align*}$$

Figure 10 shows nondimensional histories for $P_a, \dot{P}_a, F_{gas}, F_{amb}, F_{exp},$ and $F_{buoy}$ over the first 2$\frac{1}{2}$ response cycles for the benchmark test defined in Sec. IV D. We observe that $P_a$ is oscillatory while $\dot{P}_a$ increases monotonically. When $a$ is large, $P_a$ is the product of a large translational added mass and a small translational velocity; the reverse holds when $a$ is small, however, which produces rapid bubble translation at each collapse/rebound. Because $P_a$ at second collapse/rebound is essentially double that at first collapse/rebound, the second jump in translational velocity is essentially twice as large as its predecessor; similarly, the nth translational jump is essentially $n$ times as large as the first. With regard to the dilatational generalized forces, $F_{amb}$ is dominant when $a$ is large, while $F_{gas}$ and $F_{exp}$ are dominant when $a$ is small. $F_{gas}$ is a dominant force only at collapse/rebound, whereas $F_{exp}$ is a dominant force over much of a cycle. Note that $F_{exp}$ suffers extremely rapid variation as $\dot{a}$ passes through zero during collapse/rebound.

B. DAA model

From Eqs. (14), (55), and (48a) expressed at $r = a(t), \theta = \dot{\theta}(t)$, and from Eq. (47), EDAA$_1$ for dilation plus translation of a spherical surface is

$$\begin{align*}
\dot{P}_{t0} + \dot{P}_{t1} \cos \dot{\theta} &= -a^{-1}(\phi_{t0} + 2\phi_{t1} \cos \dot{\theta}) \\
- c_1^{-1}[\phi_{t0} + \phi_{t1} \cos \dot{\theta} - \phi_{t1} \dot{\theta}(\dot{\theta}) \sin \dot{\theta} \\
- (a + \dot{a} \cos \dot{\theta}) \dot{\theta}^2 - (\dot{\theta} - \dot{\theta})^2 \sin^2 \dot{\theta},
\end{align*}$$

Introducing $\dot{\theta}(\dot{\theta}) = a^{-1} \dot{\theta}(\dot{\theta})$, and Eqs. (50b) and (50c) in the right side of this equation, and then projecting the result, we obtain the modal EDAA$_1$ equations

$$\begin{align*}
\dot{\phi}_{t0} &= -a^{-1} \phi_{t0} - c_1^{-1}(\phi_{t0} - a^{-2} - \frac{1}{2}a \dot{a} - 1 \phi_{t1}), \\
\dot{\phi}_{t1} &= -2a^{-1} \phi_{t1} - c_1^{-1}(\phi_{t1} - 2a \dot{a}).
\end{align*}$$

As discussed in Sec. III B, $H_2 \phi_0$ vanishes for dilational motion. $H_2 \phi_0$ produces a $-1 \phi_{t1}$ cos $\dot{\theta}$ for translational motion, which is readily verified from Eq. (10) by noting that the pertinent solution to Laplace’s equation is $\varphi_1(r, \theta) = (r \cdot r/a) \cos \theta$. Hence, from Eq. (16), Eq. (48a) with $l = 0$ expressed at $r = a(t), \theta = \dot{\theta}(t)$, and Eq. (47) with $l = g, \text{IDA}_1$ for dilation plus translation of a spherical surface is [cf. Eq. (63)]

$$\begin{align*}
\phi_{t0} + \phi_{t1} \cos \dot{\theta} &= a^{-1} \phi_{t0} \cos \dot{\theta} \\
+ c_1^{-1}[\phi_{t0} + \phi_{t1} \cos \dot{\theta} - \phi_{t1} \dot{\theta}(\dot{\theta}) \sin \dot{\theta} \\
- (a + \dot{a} \cos \dot{\theta}) \dot{\theta}^2 - (\dot{\theta} - \dot{\theta})^2 \sin^2 \dot{\theta},
\end{align*}$$

Employing $\dot{\theta}(\dot{\theta}) = a^{-1} \dot{\theta}(\dot{\theta})$, and Eqs. (53b) and (53c) in the right side of this equation, and then projecting the result, we obtain the modal IDAA$_1$ equations [cf. Eq. (64)]

$$\begin{align*}
\phi_{t0}' &= c_1^{-1}(\phi_{t0} - a^{-2} - \frac{1}{2}a \dot{a} - 1 \phi_{t1}), \\
\phi_{t1}' &= a^{-1} \phi_{t1} + c_1^{-1}(\phi_{t1} - 2a \dot{a}).
\end{align*}$$

The substitution of $\dot{a}$ for $\phi_{t0}$ and $\ddot{a}$ for $\phi_{t1}$ in Eq. (64) yields the following DAA-consolidated kinematic-compatibility equations that correspond to Eq. (17a):

$$\begin{align*}
\dot{a} &= -a^{-1} \phi_{t0} - c_1^{-1}(\phi_{t0} - a^{-2} - \frac{1}{2}a \dot{a} - 1 \phi_{t1}), \\
\ddot{a} &= -2a^{-1} \phi_{t1} - c_1^{-1}(\phi_{t1} - 2a \dot{a}).
\end{align*}$$

The substitution of $\dot{a}$ for $\phi_{t0}$ and $\ddot{a}$ for $\phi_{t1}$ in Eq. (66) yields the following equations that correspond to (17b):

$$\begin{align*}
\dot{a} &= c_1^{-1}(\phi_{t0} - a^{-2} - \frac{1}{2}a \dot{a} - 1 \phi_{t1}), \\
\ddot{a} &= a^{-1} \phi_{t1} + c_1^{-1}(\phi_{t1} - 2a \dot{a}).
\end{align*}$$

A procedure parallel to that described between Eqs. (17) and (18) yields the following DAA-consolidated Bernoulli equations that correspond to Eqs. (18):

$$\phi_{t0} = (1 + \zeta)^{-1}\left[\frac{1}{2} \frac{1}{2} \left(\frac{\rho_g}{\rho_l} + \zeta\right) (a^2 + \frac{1}{2} \dot{a}^2) - \frac{\rho_g}{\rho_l} c_s a^{-1} \phi_{t0} + \frac{1}{2} (1 + \zeta) \dot{a} a^{-1} \phi_{t1} - Z\right],$$
\[ \phi_{g0} = (1 + \xi)^{-1}\left[ 1 + \frac{1}{2} \left( \frac{c_g}{c_l} \right) + \frac{1}{2} \xi \right] (a^2 + \frac{1}{3} \dot{u}^2) \]

\[ - c_g a^{-1} \phi_{g0} + \frac{1}{2} (1 + \xi) \dot{u} a^{-1} \phi_{g1} + \frac{c_g}{c_l} \left( 2 a^{-1} \phi_{l1} \right) \]

\[ \phi_{l1} = (1 + \xi)^{-1} \left[ 1 + \frac{\rho_g}{\rho_l} \left( 1 + 2 \xi \right) \dot{u} - \frac{\rho_g}{\rho_l} c_g \left( 2 a^{-1} \phi_{l1} \right) \right] + a^{-1} \phi_{g1} - \left[ 1 - \frac{\rho_g}{\rho_l} \right] g \dot{a} \]

\[ \phi_{g1} = (1 + \xi)^{-1} \left[ 2 + \frac{c_g}{c_l} + \xi \right] \dot{u} c_g \left( 2 a^{-1} \phi_{l1} \right) + a^{-1} \phi_{g1} + \frac{c_g}{c_l} \left[ 1 - \frac{\rho_g}{\rho_l} \right] g \dot{a} \]

where \( Z = \rho_l^{-1}(P_g - P_j + \rho_g u) + 1/3[(a^{-1} \phi_{l1})^2 - (\rho_g/\rho_l) \times (a^{-1} \phi_{g1})^2] \). The right sides of Eq. (69) are used for \( \phi_{g0}, \phi_{l1}, \phi_{g1} \), and \( \phi_{g1} \) on the right sides of Eqs. (67) and (68). Equations (67)–(69) are new; they lend themselves to numerical solution by explicit time integration with iteration keyed to \( (a^{-1} \phi_{l1})^2 \) and \( (a^{-1} \phi_{g1})^2 \). Again, there is no need to consider \( \phi_{g0} \) because Eqs. (67) and (69a), (69c), and (69d) are sufficient to calculate \( a(t) \) and \( u(t) \). Thus, we have five equations for five unknowns.

Seven initial values are needed; the first two are \( a(t_i) = a_i, \dot{a}(t_i) = \dot{a}_i \) and the second two are \( u(t_i) = 0, \dot{u}(t_i) = 0 \). To determine \( \phi_{l0}(t_i) \), we may either solve Eq. (67a) for \( \phi_{l0} \) and equate the right side of the result to the right side of Eq. (69a), or solve Eq. (68a) for \( \phi_{g0} \) and equate the right side of that result to the right side of Eq. (69b). Either way gives

\[ \phi_{l0} = -a \dot{u} \left[ 1 + \xi - \frac{1}{2} \left( 1 - \frac{\rho_g}{\rho_l} \right) \frac{\dot{a}}{c_l} + \frac{1}{2} \left( 1 - \frac{\rho_g}{\rho_l} \right) \frac{\dot{u}}{c_l} + \frac{a}{Z} \right] \]

which we must evaluate at \( t = t_i \). To determine \( \phi_{l1}(t_i) \), we may either solve Eq. (67b) for \( \phi_{l1} \) and equate the right side of the result to the right side of Eq. (69c), or solve Eq. (68b) for \( \phi_{g1} \) and equate the right side of that result to the right side of Eq. (69d). Either procedure yields

\[ \phi_{g1} = \zeta^{-1} \left[ 1 + \xi - \left( 1 - \frac{\rho_g}{\rho_l} \right) \frac{\dot{a}}{c_l} a \dot{u} - \left( 1 - \frac{\rho_g}{\rho_l} \right) \frac{g}{c_l} a^2 + 2 \phi_{l1} \right] \]

which we also must evaluate at \( t = t_i \).
In order to evaluate Eqs. (70) and (71) at \( t = t_I \), we require \( \phi_{11}(t_I) \). This may be found by first determining \( \phi_{11}(t_I) \) for the waveless model with Eqs. (58b) and (57b); the result is \( \phi_{11}(t_I) = -ga_I \). Introducing this relation into Eq. (67b) expressed at \( t = t_I \), solving for \( f_{l1}(t_I) \), and introducing the result into Eqs. (71) and (70), we obtain the initial values

\[
\begin{align*}
\phi_{11}(t_I) &= \frac{1}{2}(g/c_l)a_I^2, \\
\phi_{10}(t_I) &= -a_I \left( 1 + \frac{\rho_{gl}}{\rho_l} \frac{\dot{a}_I}{a_I} \right) \left( \frac{1 - \rho_{gl}}{\rho_l} \right) \left( \rho_{cl} \right)^{-1} \\
&\times \left( P_{gl} - P_l \right) + \frac{1}{2} \left( \frac{\rho_{gl}}{\rho_l} \right) \left( c_{gl}^2 - \frac{1}{2} c_1^{-2} \right) \left( \rho_{cl} \right)^{-1} (g a_I)^2.
\end{align*}
\]

Note that the introduction of Eqs. (72a) and (72b) into Eq. (69c) yields \( \phi_{11}(t_I) = -ga_I \), as found from the waveless model.

As mentioned at the end of Sec. III B, Eq. (72) can also be used to define the initial conditions for general bubble motion, inasmuch as deformation results only from parametric forcing [Penney and Price (1942); Ward (1943); Kolodner and Keller (1953)]. This is in contrast to dilation, which is driven by bubble pressure, and translation, which is driven by buoyancy.

Unfortunately, Eqs. (67) and (69a), (69c) and (69d) do not lend themselves to the formulation of EOM for dilation plus translation. This matters little, however, as the first-order equations integrate quite readily.

C. Computational versus experimental results for dilation plus translation

Figure 11 shows computed dilation and translation histories for a case in which experimental data pertaining to both motions exist [Snay (1962); Hicks (1970)]. The data pertain to a detonation of 227 kg of 1.5 g/cc TNT at a depth of 137 m. The histories come from the DAA-consolidated equations, Eqs. (67) and (69a), (69c), and (69d), the Hicks EOM, Eq. (59), and the DAA EOM for dilation only, Eq. (32). The solid symbols on the translational histories mark calculated bubble positions at calculated radial minima, and the circles mark measured bubble positions at measured radial minima.

Figure 11(a) shows very good agreement between the
DAA bubble-radius histories and the experimental data through the first two bubble periods; it shows satisfactory agreement between the Hicks history and the data through the first bubble period. The proximity of the two DAA histories indicates that, at this depth, the effect of translation on dilation is small. For the third bubble period, the DAA-calculated maximum radius is close to its experimental counterpart, but the associated period is too large. The comparison does not extend beyond three bubble periods because Snay characterizes the measurements of later periods as “sparse and uncertain.”

Figure 11(b) implies good DAA prediction of translation until just beyond the first bubble minimum, fair prediction from that time to just beyond the second bubble minimum, and poor prediction thereafter. It implies fair Hicks prediction until just beyond the first period and poor prediction after that. Regarding the experimental data points, Snay (1962) discusses the “rather approximate nature of the information.” To improve his prediction of translation, Hicks introduced form drag with an exaggerated drag coefficient of 2.25. We introduced a drag coefficient of 0.5 into our DAA model to produce the history marked with diamonds. The corresponding dilation history lies between the two DAA histories in Fig. 11(a).

VI. CONCLUSION

In this paper, we have introduced the following new constituents into a boundary model of an underwater-explosion bubble located well below the free surface.

1. A treatment of shock-wave-phase phenomena that relates volume acceleration to an empirical far-field pressure profile, thereby connecting the shock-wave and oscillation phases of bubble motion by means of initial conditions for the latter;

2. DAA-based response equations for oscillation-phase bubble motion that account for wave effects in both the liquid and the gas; equations are given for general motion, pure dilation of a spherical bubble, and dilation plus translation of a spherical bubble.

The result is an integrated bubble model that, when specialized to the spherical geometry, produces calculated responses in much better agreement with experimental data than those produced by waveless and mixed models.

However, as seen in Fig. 11, the DAA-based spherical model has its deficiencies. The deficiency in dilation prediction may be addressed through artificial reductions of $c_l$ and/or $c_g$ so as to fit existing bubble-maximum and bubble-period experimental data, at least through the third bubble period. The deficiency in translation prediction may then be addressed through the introduction of a drag coefficient adjusted to fit bubble-translation experimental data, although the existing data are not highly reliable.

A more satisfying strategy would employ a dilation/translation/deformation model for the oscillation phase that incorporates wave effects in both the liquid and the gas [Eqs. (17) and (18)], as well as jetting effects at bubble collapse/rebound [Best (1993); Zhang et al. (1993)]. Boundary-element bubble codes based on Eq. (12), when applied to the spherical bubble in Fig. 9, surely produce the Hicks histories in Fig. 11.

Finally, it would be worthwhile to explore the physical phenomena underlying the wave-energy sink implied in the internal doubly asymptotic approximation, Eq. (16). Bubble calculations with hydrocodes that don’t account for such energy loss overpredict bubble response as a matter of course.

ACKNOWLEDGMENTS

This work was supported under Contract No. DNA001-94-C-0004 with the Defense Threat Reduction Agency, Alexandria, VA, and Contract Nos. N00014-94-1-0796 and N00014-01-1-0154 with the Office of Naval Research, Arlington, VA. The authors thank Mr. Douglas Bruder and Mr. Michael Giltrud of DTRA, Dr. Rosby Barossoum, Dr. Luise Couchman, and Mr. Steven Schreppler of ONR, Mr. Michael Riley and Mr. Gregory Harris of the Naval Surface Warfare Center, and Mr. Austin Alvarez of Electric Boat Corporation.


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